18.642 Assignment: Problem Set 4 Fall 2024

Collaboration on homework is encouraged, but you will benefit from independent effort to solve the problems before discussing them with other people. **You must write your solution in your own words. List all your collaborators.**

1. **Brownian Motions**

   1(a) Let $B(t)$ be a standard Brownian motion. Compute
   $$E[B(t) \mid B(s)] \text{ and } Var[B(t) \mid B(s)],$$
   where $t > s \geq 0$ are fixed reals.

   1(b) Let $X(t)$ be a Brownian motion with drift μ and volatility $\sigma = 2$. Compute
   $$E[X(t) \mid X(s)] \text{ and } Var[X(t) \mid X(s)]$$

   1(c) For a standard Brownian Motion and two fixed reals $t > s > 0$, compute
   $$E[exp(\sigma(B(t) - B(s)))]$$

2. **The Brownian Scaling Relation**. Let $B(t)$ be standard Brownian Motion. Prove:

   2(a) If $B(0) = 0$, then for any $t > 0$
   $$\{B(st), s \geq 0\} = (in\ distribution)\{t^{1/2}B(s), s \geq 0.\}$$

   2(b) More generally,the two families of random variables (corresponding to stochastic processes) have the same finite dimensional distributions, that is, if $s_1 < \cdots < s_n$, then
   $$(B(s_1t, \ldots, B(s_nt)) = (in\ distribution)\ (t^{1/2}B(s_1), \ldots, t^{1/2}B(s_n))$$

3. **Sample Estimators of Diffusion Process Volatility and Drift**

Let $\{X_t\}$ be the price of a financial security that follows a geometric Brownian motion process:

$$\frac{dX(t)}{X(t)} = \mu_* dt + \sigma dW(t),$$

where

- $\sigma > 0$, is the volatility parameter
- $\mu_* \in (-\infty, \infty)$, is the drift parameter
- $dX(t)$ is the infinitesimal increment in price.
- $dW(t)$ is the increment of a standard Wiener Process, i.e, infinitesimal increments $W(t+dt) - W(t)$ are i.i.d. Normal random variables with zero mean and variance equal to 'dt'.

Consider sampling values of the price process over a fixed time period $t \in [0, T]$, at equal time increments $h = T/n$. Define

$$t_i = i \times h,\ i = 0, 1, \ldots, n$$
$$X_i = X(t_i),\ i = 0, 1, \ldots, n$$
$$Y_i = log(X_i/X_{i-1}),\ i = 1, 2, \ldots, n$$

Accept as given that:

$$Y_i \text{ are i.i.d. } N(\mu \cdot h, \sigma^2 \cdot h) \text{ random variables,}$$

(this is proven with the theory of diffusion processes/stochastic differential equations, with $\mu = \mu_* - \frac{1}{2}\sigma^2$).

3(a) Prove that the Maximum-Likelihood Estimates: $\hat{\mu}$ and $\hat{\sigma}$ for a sample: $y_1, y_2, \ldots, y_n$, are given by

$$\hat{\mu} = \frac{1}{hn}\sum_{i=1}^{n} Y_i$$
$$\hat{\sigma}^2 = \frac{1}{hn}\sum_{i=1}^{n}(Y_i - h\hat{\mu})^2$$

3(b) Derive the distribution of $\hat{\mu}$; give specific formulas for the expectation and variance of $\hat{\mu}$.

3(c) Derive the distribution of $\hat{\sigma}^2$; give specific formulas for the expectation and variance of $\hat{\sigma}^2$.

3(d) Consider increasing the number of increments n on the fixed time period $[0, T]$, and let $\hat{\mu}_n$ and $\hat{\sigma}^2_n$ be the corresponding MLEs of the parameters. Determine the limiting distributions of $\hat{\mu}_n$ and $\hat{\sigma}^2_n$.

3(e) A sequence of estimators $\hat{\theta}_n$ for a parameter θ, is weakly consistent if for any $\epsilon > 0$,

$$\lim_{n\to\infty} Pr(|\hat{\theta}_n - \theta| > \epsilon) = 0.$$

For each of $\hat{\mu}_n$ and $\hat{\sigma}^2_n$, determine whether the sequence of estimators (as the sample size n increases) is weakly consistent. (Hint: use Markov's Inequality.)

4. Consider the same process as in previous problem, but now, for fixed values of μ and σ, consider sampling n values of the price process over a fixed time period $t \in [0, T]$, at variable increments $h_i > 0$, $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} h_i = T$. Define

$$t_i = \sum_{j=1}^{i} h_j,\ i = 0, 1, \ldots, n$$
$$X_i = X(t_i),\ i = 0, 1, \ldots, n$$
$$Y_i = log(X_i/X_{i-1}),\ i = 1, 2, \ldots, n$$

Accept as given that:

Y_i are i.i.d. $N(\mu \cdot h_i, \sigma^2 \cdot h_i)$ random variables,

(this is proven with the theory of diffusion processes/stochastic differential equations).

3(a) Derive the MLE for μ and its distribution for a fixed set of sampling increments $\{h_i\} : \sum_{i=1}^{n} h_i = T$.

3(b) Derive the MLE for σ^2 and its distribution for a fixed set of sampling increments $\{h_i\} : \sum_{i=1}^{n} h_i = T$.

3(c) If limited to sampling $n + 1$ price points of $\{X_t\}$, (including X_0 and X_T) prove that

- For estimating σ^2, sampling, the ML estimators vary with the increment spacing, but the variance of these estimators are all equal, regardless of the increment spacing.
- For estimating μ, all ML estimators are the same and have the same variance, regardless of the increment spacing.

5. **Covariance Stationary AR(2) Processes**

Suppose the discrete-time stochastic process $\{X_t\}$ follows a second-order auto-regressive process $AR(2)$:

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \eta_t,$$

where $\{\eta_t\}$ is $WN(0, \sigma^2)$, with $\sigma^2 > 0$, and ϕ_0, ϕ_1, ϕ_2, are the parameters of the autoregression.

(a) If $\{X_t\}$ is covariance stationary with finite expectation $\mu = E[X_t]$ show that

$$\mu = \frac{\phi_0}{1-\phi_1-\phi_2}$$

(b) For the autocovariance function

$$\gamma(k) = Cov[X_t, X_{t-k}],$$

show that

$$\gamma(k) = \phi_1\gamma(k-1) + \phi_2\gamma(k-2), \text{ for } k = 1, 2, \ldots$$

(c) For the autocorrelation function

$$\rho_k = corr[X_t, X_{t-k}],$$

show that

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \text{ for } k = 1, 2, \ldots$$

(d) **Yule-Walker Equations**

Define the two linear equations for ϕ_1, ϕ_2 in terms of ρ_1, ρ_2 given by $k = 1, 2$ in (c):

$$\begin{aligned} \rho_1 &= \phi_1\rho_0 + \phi_2\rho_{-1} \\ \rho_2 &= \phi_1\rho_1 + \phi_2\rho_0 \end{aligned}$$

Using the facts that $\rho_0 = 1$, and $\rho_k = \rho_{-k}$, this gives

$$\begin{aligned} \rho_1 &= \phi_1 \quad + \quad \phi_2\rho_1 \\ \rho_2 &= \phi_1\rho_1 \quad + \quad \phi_2 \end{aligned}$$

which is equivalent to:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Solve for ϕ_1 and ϕ_2 in terms of ρ_1, and ρ_2.

(e) Solve for ρ_1, and ρ_2 in terms of ϕ_1 and ϕ_2.

6. **Autoregressive Moving Average Process:** $ARMA(1,1)$

Suppose the discrete stochastic process $\{X_t\}$ follows a covariance stationary ARMA(1,1) model:

$$\begin{aligned} X_t - \phi_1 X_{t-1} &= \phi_0 + \eta_t + \theta_1\eta_{t-1} \\ (1 - \phi_1 L)X_t &= \phi_0 + (1 + \theta_1 L)\eta_t, \quad t = 1, 2, \ldots \quad (\eta_0 = 0) \end{aligned}$$

where $\{\eta_t\}$ is $WN(0, \sigma^2)$.

(a) Prove that

$$\mu = E[X_t] = \frac{\phi_0}{1-\phi_1}$$

(b) Prove that

$$\sigma_X^2 = Var(X_t) = \gamma_0 = \frac{\sigma^2[1+\theta_1^2+2\phi_1\theta_1]}{1-\phi_1^2} = \sigma^2\left[1+\frac{(\theta_1+\phi_1)^2}{1-\phi_1^2}\right]$$

(c) Prove that the auto-correlation function (ACF) of the covariance stationary $ARMA(1,1)$ process is given by the following recursions:

$$\begin{aligned} \rho_1 &= \phi_1 + \frac{\theta_1\sigma^2}{\gamma_0} \\ \rho_k &= \phi_1\rho_{k-1}, \quad k > 1 \end{aligned}$$

(d) Compare the ACF of the $ARMA(1,1)$ process to that of the $AR(1)$ process with the same parameters ϕ_0, ϕ_1.

What pattern in the ACF function of an $ARMA(1,1)$ model is not possible with an $AR(1)$ model? Suppose an economic index time series follows such an $ARMA(1,1)$ process. What behavior would it exhibit?

(e) Compare the ACF of the $ARMA(1,1)$ process to that of the $MA(1)$ process with the same parameters ϕ_0, θ_1.

What is the simple pattern of the ACF function for an $MA(1)$ process. How does this pattern change for an $MA(q)$ process, with $q > 1$?

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