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Thue's Theorem.# $\{a, b, c \mid a, b, c \in \mathbb{Z} \text{ nonzero} \mid ax^3 + by^3 = c\}$ is finite.

$$\begin{array}{c} \updownarrow \\ x^3 - by^3 = c \quad b, c > 0. \end{array}$$

$$x^3 - by^3 = (x - \beta y) \underbrace{(x^2 + \beta xy + \beta^2 y^2)}_{\geq \frac{3}{4} \beta^2 y^2} \quad \beta = \sqrt[3]{b}.$$

$$\left| \frac{x}{y} - \beta \right| \leq \frac{4|c|}{3\beta^2} \frac{1}{|y|^3}$$

DAT:

$$\beta = \sqrt[3]{b} \notin \mathbb{Q}.$$

$$\# \left\{ p, q \in \mathbb{Z} \mid q > 0, \left| \frac{p}{q} - \beta \right| \leq \frac{C}{q^3} \right\} < \infty \quad \forall C.$$

(1) $F(x, y) \in \mathbb{Z}[x, y]$ that vanishes to high order at (β, β) .(2) Upper Bound for $|F(\frac{p_1}{q_1}, \frac{p_2}{q_2})|$ in terms of $|\frac{p_1}{q_1} - \beta|$ and $|\frac{p_2}{q_2} - \beta|$.(3) Lower bound for $|F(\frac{p_1}{q_1}, \frac{p_2}{q_2})|$ Recall APT

$$\beta = \sqrt[3]{b} \quad m, n \in \mathbb{Z} \mid m+1 > \frac{2}{3}n \geq m \geq 3.$$

 $\exists F(x, y) \in \mathbb{Z}[x, y] \mid F(x, y) = P(x) + YQ(x)$

$$= \sum_{i=0}^{m+n} u_i x^i + v_i x^i y.$$

$$F^{(k)}(\beta, \beta) = 0 \quad 0 \leq k \leq n \quad \max_{0 \leq i \leq m+n} \{|u_i|, |v_i|\} \leq 2(16b)^{m+n}$$

Recall

Sth:

$$\exists c_1(b) > 0 \text{ s.t. } |F^{(t)}(x,y)| \leq c_1^n \{ |x-p|^{n-t} + |y-p| \}$$

for all real x,y s.t. $|x-p| \leq 1, \forall t \leq n$.

Non Vanishing Theorem NVT. (Today).

$$\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q} \text{ in lowest terms.}$$

$$\exists c_2(b) > 0 \text{ and } t \in \mathbb{Z} \text{ s.t. } 0 \leq t \leq 1 + \frac{c_2 n}{\log q_1},$$
$$F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \neq 0.$$

$$T \equiv \text{largest integer s.t. } F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) =$$

$$P^{(t)}\left(\frac{p_1}{q_1}\right) + Q^{(t)}\left(\frac{p_1}{q_1}\right)\left(\frac{p_2}{q_2}\right) = 0 \quad 0 \leq t < T$$

eliminate $\frac{p_2}{q_2}$ from each pair of equations. s,t

$$P^{(t)}\left(\frac{p_1}{q_1}\right) Q^{(s)}\left(\frac{p_1}{q_1}\right) - P^{(s)}\left(\frac{p_1}{q_1}\right) Q^{(t)}\left(\frac{p_1}{q_1}\right) = 0$$

$$0 \leq s, t < T.$$

$$W_{P,Q}(x) = \begin{vmatrix} P(x) & Q(x) \\ P'(x) & Q'(x) \end{vmatrix} = P(x) Q'(x) - P'(x) Q(x).$$

$$W^{(r)}(x) = \sum_{i+j=r} \frac{i! j!}{r!} \left(P^{(i)}(x) Q^{(j+1)}(x) - Q^{(i)}(x) P^{(j+1)}(x) \right)$$

$$x = \frac{p_1}{q_1} \quad r < T-1 \quad W^{(r)}\left(\frac{p_1}{q_1}\right) = 0 \quad \forall 0 \leq r \leq T-1$$

$(x - \frac{p_1}{q_1})^{T-1}$ is a factor of $W(x)$.

Gauss' Thm.

$P \in \mathbb{Z}[x]$ factors in $\mathbb{Q}[x]$
 \Rightarrow factors in $\mathbb{Z}[x]$.

$$W(x) = (q_1 x - p_1)^{T-1} V(x) \quad V(x) \in \mathbb{Z}[x].$$

$$P(x) = \sum u_i x^i \quad Q(x) = \sum v_i x^i$$

$$W(x) = \sum_{ij} j (u_i v_j - v_i u_j) x^{i+j-1}$$

$$\max_{ij \leq m+n} |j (u_i v_j - v_i u_j)| \leq 2(m+n) \left(\max_{i \leq m+n} \{u_i, v_i\} \right)^2$$

$$\leq 2(m+n) \left(2 \cdot (16b)^{q(m+n)} \right)^2 \quad m \leq \frac{2}{3}n.$$

$$\leq C_3^n \quad C_3 \text{ is a function of } b.$$

$$q_1^{T-1} \leq \text{largest coefficient of } W(x) \leq C_3^n.$$

$$T \leq 1 + \frac{C_2 n}{\log q_1} \quad C_2 = \log C_3.$$

$$\exists t \mid 0 \leq t \leq 1 + \frac{C_2 n}{\log q_1} \quad F^{(x)} \left(\frac{p_1}{q_1}, \frac{p_2}{q_2} \right) \neq 0.$$

So done except what if $W(x) = 0$?

Assume that $W(x) = 0$.

$$\Rightarrow P(x)Q'(x) = Q(x)P'(x)$$

$$\Rightarrow \frac{d}{dx} \left(\frac{P(x)}{Q(x)} \right) = 0.$$

$$P(x) = aQ(x). \quad a \in \mathbb{Q}.$$

$$\text{Then } F(x, y) = (a+y)Q(x)$$

$$F^{(k)}(\beta, \beta) = \underbrace{(a+\beta)}_{\neq 0} Q^{(k)}(\beta) = 0. \quad 0 \leq k < n.$$

$$\Downarrow \\ = 0.$$

$$(x-\beta)^n \mid Q(x).$$

$$(x^2-b)^n \mid Q(x) \quad \deg Q(x) \geq 3n.$$

$$\text{but it isn't: } \deg Q \leq m+n \leq \frac{5}{3}n.$$

$$\text{So } W(x) \neq 0.$$