10/8/04

The height of $P+P_0$

<u>Lemma 2</u>. Let P_0 be a fixed rational point on C.
$\exists\, \kappa_0$, depending on P_0 and a, b, c s.t.
$$h(P+P_0) \leq 2h(P) + \kappa_0 \quad \forall P \in C(\mathbb{Q}).$$

Let $P_0 = (x_0, y_0)$ $P = (x, y)$
If $P \in C(\mathbb{Q})$, then x and y have the following form:
$x = \frac{m}{e^2}$, $y = \frac{n}{e^3}$.

$m, n, e \in \mathbb{Z}$ $e > 0$
$\gcd(m, e) = \gcd(n, e) = 1$.

Proof: Suppose $x = \frac{m}{M}$, $y = \frac{n}{N}$, both in lowest terms.
Substitute $\left(\frac{m}{M}, \frac{n}{N}\right)$ into $y^2 = x^3 + ax^2 + bx + c$

$$M^3n^2 = N^2m^3 + aN^2Mm^2 + bN^2M^2m + cN^2M^3.$$
$N^2 \mid M^3n^2$, and since $\gcd(N, n) = 1$
$N^2 \mid M^3$.

$$M^2n^2 = \frac{N^2m^3}{M} + aN^2m^2 + bN^2Mm + cN^2M^2$$

$$M \mid N^2m^3, \gcd(M, m) = 1 \longrightarrow M \mid N^2$$

$$Mn^2 = \frac{N^2m^3}{M^2} + \frac{aN^2m^2}{M} + bN^2m + cN^2M$$

$M^2 \mid N^2 m^3$ so $M^2 \mid N^2 \Rightarrow M \mid N$.

(One more time this process: divide equation by M again).
will get $M^3 \mid N^2$.

$$M^3 = N^2.$$

If $e = N/M$ then $e^2 = \frac{N^2}{M^2} = M \quad e^3 = \frac{N^3}{M^3} = N$.

$P = \left(\frac{m}{e^2}, \frac{n}{e^3}\right) \in C(\mathbb{Q})$

Height of a rational pt is height of x-coordinate

so

$H(P) = \frac{m}{e^2} = \max(|m|, |e^2|)$

$|m| \le H(P)$

$|e^2| \le H(P)$.

The numerator n can also be bound in terms of $H(P)$

$\exists$ k s.t. $|n| \le K H(P)^{3/2}$.

substitute $\left(\frac{m}{e^2}, \frac{n}{e^3}\right)$ into C

$n^2 = m^3 + ae^2m^2 + be^4m + ce^6$

$|n^2| \le |m^3| + |ae^2m^2| + |be^4m| + |ce^6|$.

$|n^2| \le H(P)^3 + |a| H(P)^3 + |b| H(P)^3 + |c| H(P)^3$

$|n| \le H(P)^{3/2} \sqrt{1+|a|+|b|+|c|}$

K.

$$H(\xi) \leq \max \{ |Ane + Bm^2 + Cme + De^4|, \; |En^2 + Fme^2 + Ge^4| \}.$$

recall
$$e^2 \leq H(P)$$
$$e \leq H(P)^{\frac{1}{2}}$$
$$n \leq K H(P)^{\frac{3}{2}}$$
$$m \leq H(P).$$

numerator:
$$|Ane^2 + Bm^2 + Cme^2 + De^4| \leq$$
$$|Ane^2| + |Bm^2| + |Cme^2| + |De^4|.$$
$$\leq ((|AK| + |B| + |C| + |D|) H(P)^2$$

Denominator:
$$|En^2 + Fme^2 + Ge^4| \leq |Em^2| + |Fme^2| + |Ge^4|$$
$$\leq (|E| + |F| + |G|) H(P)^2).$$

$$H(\xi) = H(P + P_0) \leq \cancel{\max}$$
$$\max \{ |AK| + |B| + |C| + |D|, \; |E| + |F| + |G| \} H(P)^2$$

log of both sides:

$$h(P + P_0) \leq 2h(P) + k_0$$

where $k_0 = \log \max \{ \quad \}$ depends only on a, b, c and P_0.

$P \notin \{P_0, -P_0, O\}$, $P_0 \notin \{O\}$

$P = (x, y)$ $\quad P_0 = (x_0, y_0)$

$P + P_0 = (\xi, \eta)$

$H(P+P_0) = H(\xi)$. – find formula for ξ.

$$\xi = \lambda^2 - a - x - x_0, \quad \lambda = \frac{y - y_0}{x - x_0}.$$

$$\xi = \frac{(y-y_0)^2}{(x-x_0)^2} - a - x - x_0$$

$$= \frac{(y-y_0)^2 - (x-x_0)^2(x + x_0 + a)}{(x-x_0)^2}$$

$$y^2 = x^3 + ax^2 + bx + c$$

$$\xi = \frac{Ay + Bx^2 + Cx + D}{Ex^2 + Fx + G}$$

where $A \ldots G$ are integers. in terms of a, b, c as well as (x_0, y_0)

Substitute $P = \left(\frac{m}{e^2}, \frac{n}{e^3}\right)$,

$$\xi = \frac{Ane + Bm^2 + Cme^2 + De^4}{Em^2 + Fme^2 + Ge^4}$$