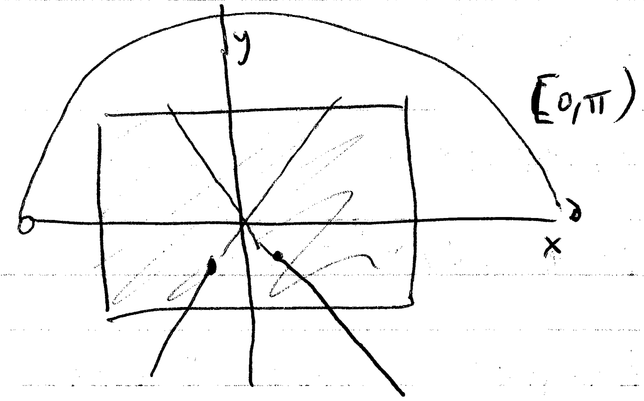
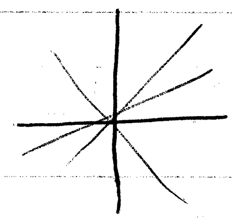
Thomas Coffee 9/13/04



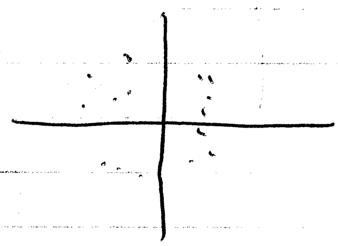
$\mathbb{A}^3$



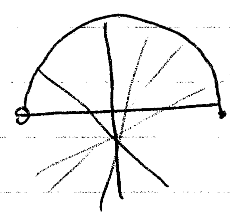
$[a,b,c]$

$(a,b,c) \sim (ta,tb,tc) \quad t \neq 0.$

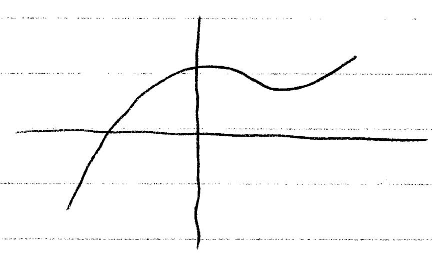
$\mathbb{A}^2$



$\mathbb{P}^1$



$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$



$f(x,y) = 0$

$F(X,Y,Z) = 0.$

$F(X,Y,Z) \overset{=0}{} \Longleftrightarrow F(tX,tY,tZ) = 0$

$t \neq 0.$

$F(X,Y,Z) = 0 \qquad F(X,Y,Z) = \sum_{i,j,k} a_{ijk} X^i Y^j Z^k$

$F(tX,tY,tZ) = \sum_{i,j,k} a_{ijk} t^{i+j+k} X^i Y^j Z^k$

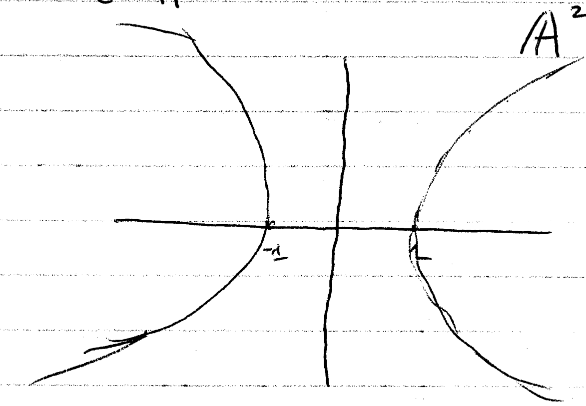
$F(X,Y,Z) = n \cdot F(tX,tY,tZ).$

$i+j+k=d$

<u>homogeneous</u>

$X^2Y - Z^3 + 2XY^2 = 0$

Affine part $\in \mathbb{A}^2$ + infinite part $\in \mathbb{P}^1$



$X^2 - Y^2 - Z^2 = 0$

$X^2 - Y^2 - 1 = 0$

$X = Y$

$X = -Y$.

$[1,1]$

$[1,-1]$

Mead

Jacob Fox 9/13/04

Projective Curve
$C: F(X,Y,Z) = 0$

Affine Curve $C_0: f(x,y) = F(X,Y,1) = 0.$

Points at infinity: points on C with $Z = 0$
correspond to limiting directions to tangent lines of C_0

Dehomogenization: Going from homogeneous $F(X,Y,Z)=0$
to inhomogeneous $f(x,y)$.

$$F(X,Y,Z) = 3X^2Y + Y^3 - YZ^2 + Z^3$$
$$f(x,y) = 3x^2y + y^3 - y + 1$$

$$C_0: f(x,y) = \sum a_{ij} x^i y^j = 0$$

$$\deg f = \max_{a_{ij} \neq 0} i+j$$

$f(x,y) = x^6y^2 + y^9 + 2xy^7 = 0$ $\deg(f) = 9.$

$F(X,Y,Z) = \sum a_{ij} X^i Y^j Z^{d-i-j}$ $d = \deg f.$

(1) F is homogeneous of degree d.
(2) dehomogenization of F is f.
(3) $F(X,Y,0)$ is not identically 0.

$$f(x,y) = x^3 + x^2y^2 - 7xy$$
$$F(X,Y,Z) = X^3Z + X^2Y^2 - 7XYZ^2$$

$F(x,y,z) = x^3y - 2x^2y^2 + z^4 = 0$ $[2,1,0]$

$F(x,y,1) = x^3y - 2x^2y^2 + 1 = 0.$

$F(x,1,z) = x^3 - 2x^2 + z^4 = 0$

$[2,0]$

$F(1,y,z) = \cancel{x}y - 2y^2 + z^4$ $[\frac{1}{2},0]$

Classical Algebraic Geometry: Solutions in $\mathbb{C}$.

Number Theory: Solutions in $\mathbb{Z}$ or $\mathbb{Q}$.

$C: F(x,y,z) = 0$

$F(x,y,z) = \sum a_{iju}\, x^i y^j z^{d-i-j}$

$cF(x,y,z) = 0 \Leftrightarrow F(x,y,z).$

$\frac{1}{2}xy - \frac{1}{3}x^2 + \frac{1}{4}z^2 = 0$

$6xy - 4x^2 + 3z^2 = 0$

$C(\mathbb{Q}) = \{[a,b,c] \in \mathbb{P}^2 : a,b,c \in \mathbb{Q},\ F(a,b,c) = 0\}.$

$[1,2,3] \in C \Rightarrow [1,2,3] \in C(\mathbb{Q}).$

$\left[\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{\sqrt{2}}\right]$

$a,b,c \in C(\mathbb{Q}) \Leftrightarrow [ta,tb,tc]$ ~~a~~'s rational and $[a,b,c] \in C.$

~~$C(\mathbb{Z}) \subset C(\mathbb{Q})$~~ (erased)

$C(\mathbb{Z}) = C(\mathbb{Q})$

~~$[a_1, b_1, c]$~~

$C_0(\mathbb{Z}) \neq C_0(\mathbb{Q})$

$C_0(\mathbb{Z}) = \{(r,s) : f(r,s) = 0,\ r, s \in \mathbb{Z}\}$.

$x^2 + y^2 = 1$ $\quad \left(\frac{3}{5}, \frac{4}{5}\right)$ $\left(\frac{5}{13}, \frac{12}{13}\right)$

$(\pm 1, 0)$ $(0, \pm 1)$

Mead