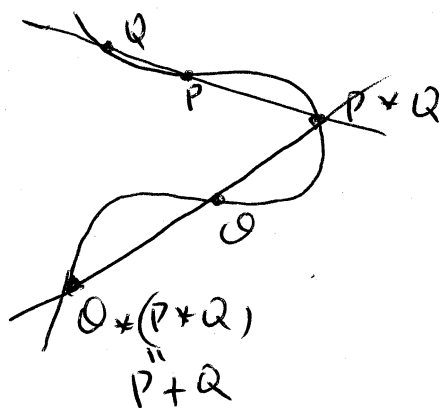


Weierstrass normal form



Mordell's Theorem:

The ~~set of~~ group of rational points on a nonsingular cubic curve is finitely generated.

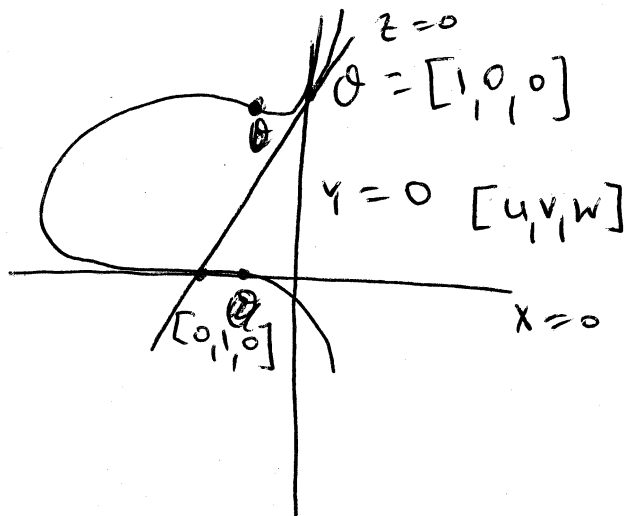
Def. A cubic curve is in Weierstrass normal form (WNF) if it has the form

$$v^2 = 4u^3 - g_2 u - g_3$$

Or more generally,

$$y^2 = x^3 + ax^2 + bx + c.$$

We'll show that any cubic can be transformed into WNF by a transformation taking points in $\mathbb{Q}^2 \rightarrow \mathbb{Q}^2$



$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Def. Such a transformation is a projective transformation if this matrix is invertible.

After transforming any cubic curve by a projective transf., we still have a cubic

$$C: AX^3 + BX^2Y + CY^2 + DY^3 + EX^2Z + FXYZ + GY^2Z + HXZ^2 + IYZ^2 + JZ^3 = 0$$

$$= f(X, Y, Z)$$

$$[1, 0, 0] \in C' \Rightarrow A = 0$$

$$[0, 1, 0] \in C' \Rightarrow D = 0.$$

$$\frac{\partial f}{\partial Z}(1, 0, 0) = 0 \Rightarrow E = 0$$

$$\frac{\partial f}{\partial X}(0, 1, 0) = 0 \Rightarrow C = 0.$$

$$x_1 y_1^2 + (ax_1 + b)y_1 = cx_1^2 + dx_1 + e$$

multiply through by x_1 :

$$x_1^2 y_1^2 + (ax_1 + b)x_1 y_1 = cx_1^3 + dx_1^2 + ex_1$$

$$\text{let } x_1 y_1 = y_2:$$

$$y_2^2 = (ax_1 + b)y_1 = cx_1^3 + dx_1^2 + ex_1$$

$$\text{let } y_2 = y_3 - \frac{1}{2}(ax_1 + b)$$

$$y_3^2 + \frac{(ax_1 + b)^2}{2} = cx_1^3 + dx_1^2 + ex_1$$

$$x_1 = cx_1 \quad y_3 = c^2 y$$

$$c^4 y^2 = \frac{(ax+b)^2}{2} = c^4 x^3 - dc^2 x^2 + ecx$$

Cancel c^4 and rearrange

$$y^2 = x^3 + \left(\frac{a}{c^2} - \frac{a^2 c^2}{4} \right) x^2 + (ec - bx) - \frac{b^4}{4}$$

Example. $u^3 + v^3 = \alpha \quad \alpha \in \mathbb{Q}$.

Homog: $u^3 + v^3 - \alpha w^3 = 0$ contains $[1, -1, 0]$.

$$x = \frac{12\alpha}{u+v}, \quad y = 3b\alpha \frac{u-v}{u+v}$$

$$\Rightarrow y^2 = x^3 - 432\alpha^2 \quad (\text{in WNF})$$

Inverting,

$$u = \frac{3b\alpha + y}{6x}, \quad v = \frac{3b\alpha - y}{6x}$$

$\alpha = 6$ Then $\left(\frac{17}{21}, \frac{37}{21} \right)$ is on the curve.

$$\lambda = \frac{12\alpha}{u+v} = 28$$

$$y = 3b\alpha \frac{u-v}{u+v} = 80$$

And in fact, $80^2 = 28^3 - 432 \cdot 6^2$