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Thue's Theorem. Let a, b, c be nonzero integers. Then the equation $ax^3 + by^3 = c$ has only finitely many solutions.

Diophantine Approximation Thm: Let b be a positive integer that is not a perfect cube, and let $\beta = \sqrt[3]{b}$. Let C be a fixed positive constant. Then there are finitely many pairs of integers (p, q) with $q > 0$ which satisfy

$$\left| \frac{p}{q} - \beta \right| \leq \frac{C}{q^3} \quad *$$

$$x^3 - by^3 = c \quad b, c > 0$$

$$\left| \frac{x}{y} - \beta \right| \leq \frac{C}{|y|^3} \quad C = \frac{4|c|}{3\beta^3}$$

$$x^3 - by^3 = (x - \beta y)(x^2 + \beta yx + \beta^2 y^2)$$

$$p^3 - bq^3 = (p - \beta q)(p^2 + \beta pq + \beta^2 q^2)$$

$$\frac{1}{q^3} \leq \left| \frac{p^3 - bq^3}{q^3} \right| = \left| \frac{p}{q} - \beta \right| \left| \left(\frac{p}{q} \right)^2 + \beta \frac{p}{q} + \beta^2 \right|$$

From (*), $\frac{p}{q} \leq \beta + \frac{C}{q^3} \leq \beta + C$

so $\rightarrow \leq \left| \frac{p}{q} - \beta \right| |3\beta^2 + 3\beta C + C^2|$.

$$C' = 3\beta^2 + 3\beta C + C^2$$

$$\left| \frac{p}{q} - \beta \right| \geq \frac{1}{C' q^3}$$

Suppose this held for a smaller exponent:

$$\frac{C}{q^3} \geq \left| \frac{p}{q} - \beta \right| \geq \frac{1}{C' q^{2.9}}$$

$$(CC')^{10} \geq q$$

$$\beta - \frac{C}{q^3} \leq \frac{p}{q} \leq \beta + \frac{C}{q^3}$$

$$\beta - C \leq \beta q - \frac{C}{q^3} \leq p \leq \beta q + \frac{C}{q^3} \leq \beta (CC')^{10} + C$$

$$|F(x)| = |x^3 - b| = (x - \beta)(x^2 + x\beta + \beta^2)$$

$$\left| F\left(\frac{p}{q}\right) \right| \geq \frac{1}{q^3}$$

$$\left| \frac{p}{q} - \beta \right| C''$$

$$F(x) = (x - \beta)^n G(x)$$

$$\frac{1}{q^d} \leq \left| F\left(\frac{p}{q}\right) \right| \leq C'' \left| \frac{p}{q} - \beta \right|^n$$

integer coefficients.

some

C'' independent of p, q

$d = \text{degree of } F.$

we'd like to have $d \leq 3n.$

but can't since $(x^3 - b)^n \mid F(x).$ $d = \deg F(x) \geq 3n.$

New approach.

$$F(x, y) \in \mathbb{Z}[x, y]$$

F to vanish to higher order at (β, β) . and
compare upper + lower bounds on

$$F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \quad \frac{p_1}{q_1}, \frac{p_2}{q_2} \text{ both satisfy } *.$$

1. Find a good polynomial $F(x, y) \in \mathbb{Z}[x, y]$.
2. Derive an upper bound on $\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right|$ in terms of $\left|\frac{p_1}{q_1} - \beta\right|$ and $\left|\frac{p_2}{q_2} - \beta\right|$.
3. Derive a lower bound on $\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right|$. In particular, show that it is nonzero.

More details. Suppose \exists infinitely many rationals satisfying $*$.

Pick $\frac{p_1}{q_1}$ that satisfies $*$ with q_1 large. Then pick $\frac{p_2}{q_2}$ that satisfies $*$ with q_2 is much larger than q_1 .

Consider $\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right|$ and conclude that it's very small.

Step 3. The auxiliary polynomial F does not vanish at $\frac{p_1}{q_1}, \frac{p_2}{q_2}$.

$$\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| = \left| \frac{\text{nonzero-integer}}{q_1^d q_2^e} \right| \quad d, e \text{ degrees in } x, y$$
$$\geq \frac{1}{q_1^d q_2^e}$$