11/29/04 Geoffrey Ligongo

Recall APT from last time:
Let b be an integer and $\beta = \sqrt[3]{b}$ and m, n satisfy $m+1 \geq \frac{2}{3}n \geq m \geq 3$.
Then $\exists$ a polynomial with integer coefficients
$$F(x,y) = P(x) + y\,Q(x) = \sum_{i=0}^{m+n} u_i x^i + v_i x^i Y \quad \text{s.t.}$$

$$F^{(k)}(\beta,\beta) = 0 \text{ for all } 0 \leq k \leq n, \quad \max_{0 \leq i \leq m+n} \{|u_i|, |v_i|\} \leq 2 \cdot (6b)^{9(m+n)}$$

Smallness Theorem

Let $F(X,Y)$ be APT polynomial. $\exists$ constant $c_1 > 0$, depending only on b, such that $\forall x, y$: $|x-\beta| \leq 1$ and any integer $0 \leq t \leq n$

$$|F^{(t)}(x,y)| \leq c_1^n \{ |x-\beta|^{n-t} + |y-\beta| \}.$$

At (β,β), partial derivative of F vanish, so
$$F(x,y) = P(x) + Y\,Q(x).$$

$$= \text{(taylor expansion about } (\beta,\beta)\text{)}$$

$$\sum_{k,j \in \mathbb{N}} \frac{1}{k!\,j!} \frac{\partial^{k+j} F}{\partial X^k Y^j}(\beta,\beta) \cdot (X-\beta)^k (Y-\beta)^j$$

$$= \sum_{k=0}^{m+n} F^{(k)}(\beta,\beta)(x-\beta)^k + \sum_{k=0}^{m+n} Q^{(k)}(\beta)(x-\beta)^k(Y-\beta)$$

$F^{(k)}(\beta,\beta) = 0$ for all $0 \leq k \leq n$, so

$$F(x,Y) = \sum_{k=n}^{m+n} F^{(k)}(\beta,\beta)(x-\beta)^k + \sum_{k=0}^{m+n} Q^{(k)}(\beta)(x-\beta)^k(Y-\beta$$

To find $F^{(t)}(X,Y)$ differentiate t times, then divide by $t!$

$$F^{(t)}(X,Y) = \sum_{k=n}^{m+n} F^{(k)}(\beta,\beta)\binom{k}{t}(X-\beta)^{k-t}$$
$$+ \sum_{k=0}^{m+n} Q^{(k)}(\beta)\binom{k}{t}(X-\beta)^{k-t}(Y-\beta)$$

$$= \left\{ \sum_{k=n}^{m+n} F^{(k)}(\beta,\beta)\binom{k}{t}(X-\beta)^{k-n} \right\}(X-\beta)^{n-t}$$
$$+ \left\{ \sum_{k=0}^{m+n} Q^{(k)}(\beta)\binom{k}{t}(X-\beta)^{k-t} \right\}(Y-\beta).$$

because $(X-\beta)^{n-t}$ and $(Y-\beta)$ are factors, values of X, Y close to β make $F^{(t)}(X,Y)$ very small.

Let $X=x$ and $Y=y$ and use triangle inequality.

$$|F^{(t)}(x,y)| \le \left\{ \sum |F^{(k)}(\beta,\beta)| \binom{k}{t} |x-\beta|^{k-n} \right\} |x-\beta|^{n-t}$$
$$+ \left\{ \sum |Q^{(k)}(\beta)| \binom{k}{t} |x-\beta|^{k-t} \right\} |y-\beta|$$

$|x-\beta| \le 1$ by assumption

$\binom{k}{t}|x-\beta|^e \le 2^{m+n}$ for any $k \le m+n$, any $e \ge 0$.

$$|F^{(t)}(x,y)| \le C_1 \left\{ |x-\beta|^{n-t} + |y-\beta| \right\}.$$

$$|F^{(k)}(\beta,\beta)| = \left| \sum_{i=k}^{m+n} \binom{i}{k} \left(u_i \beta^{i-k} + v_i \beta^{i-k+1} \right) \right|$$

$$\le (m+n+1) \cdot \max_{0\le i\le m+n} \binom{i}{k} \cdot 2 \max_{0\le i\le m+n} \{|u_i|, |v_i|\} \beta^{m+n}$$

$k \le m+n.$

$$\sum_{k=n}^{m+n} |F^{(k)}(\beta,\beta)| \binom{k}{t} |x-\beta|^{k-n} \le (m+1) \cdot 4 \left(2^{38} b^{28/3}\right)^{m+n} \cdot 2^{m+n}$$

$m+1 = \#$ of terms in the sum.

because $m \le \frac{2}{3} n$

$$\le \left(2^{42} b^{28/3}\right)^{m+n} \le \left(2^{70} b^{140/9}\right)^{n}$$

$$|Q^{(k)}(\beta)| = \left| \sum_{i=k}^{m+n} \binom{i}{k} v_i \beta^{i-k} \right| \le (m+n+1) 2^{m+n} \cdot \max_{0\le i\le m+n} |v_i| \beta^{m+n}$$

$$\le 2^{2(m+n)} \cdot 2 (16b)^{9(m+n)} \cdot b^{(m+n)/3}.$$

$$= 2\left(2^{38} b^{28/3}\right)^{m+n}$$

$$\sum_{k=0}^{m+n} |Q^{(k)}(\beta)| \binom{k}{t} |x-\beta|^{k-t} \le$$

$$(m+n+1) 2 \left(2^{38} b^{28/3}\right)^{m+n} \cdot 2^{m+n}$$

$$\le \left(2^{41} b^{28/3}\right)^{m+n} \le \left(2^{205/3} b^{140/9}\right)^{n}$$

$$|F^{(t)}(x,y)| \le c_1 \{ |x-\beta|^{n-t} + |y-\beta| \}, \quad c_1 = 2^{70} b^{140/9}$$