Lecture 1: Introduction, Simple and Semisimple Modules, Skew Fields

1 February 7 - Introduction, simple and semisimple modules, skew fields

Noncommutative algebra studies algebraic phenomena that arise in a variety of contexts in mathematics and physics, wherever one encounters a multiplication rule where the commutativity law $ab = ba$ fails. An example familiar from linear algebra is multiplication of matrices. Noncommutative groups and Lie algebras also come with such a multiplication; we will also require an addition law compatible with multiplication via the distributive law, groups and Lie algebras can be fit into that framework by passing to the group ring and enveloping algebra respectively.

Some approaches to noncommutative algebras are inspired by known results about commutative ones, where we have familiar concepts of radical, localization etc. We will see noncommutative analogues of these concepts later in the course. Another way to relate the noncommutative and the commutative settings is by deforming a commutative multiplication to obtain a noncommutative one; due to its relation to quantum physics this procedure is sometimes called quantization.

Just as in the case of groups or commutative algebras, much of the work with the abstract composition rule comprising the structure of a group or a ring involves realizing it as composition of actual symmetries of a specific set or abelian group, this leads to the concept of an action of a group on a set, and an action of a ring on a module. A noncommutative ring will be the main protagonist of our story, the plot develops as the protagonist acts (on a module)!

The language of categories and functors is ubiquitous in modern algebra, including study of noncommutative rings and modules over them. Its general concepts and their application to rings and modules will be discussed in the lectures.

Powerful tools for study of rings and modules come from homological algebra, we will introduce its basic concepts in the course.

We will look into core topics in noncommutative ring theory such as polynomial identities and rate of growth of algebras, and also touch upon connections of noncommutative algebra to other areas such as number theory (Brauer groups), Lie theory (Amitsur-Levitzki Theorem, Goldie rank) etc.

The course ends with a brief discussion of noncommutative geometry, an area that grew out of an attempt to connect noncommutative algebra to geometry inspired by the success of algebraic geometry which provides such a connection for commutative algebra.

1.1 Rings, Modules, Ideals

Passing to formal math, the main object of study for us will be associative, possibly noncommutative rings.

Definition 1.1: *A **ring** $(R, +)$ is an abelian group with a multiplication that is associative and distributes over addition. Unless stated otherwise, rings will be unital (have a multiplicative identity). Homomorphisms of rings are required to send* 1 *to* 1.

Remark 1.2: Associativity is equivalent to the fact that left multiplication commutes with right multiplication.

Definition 1.3: *Let R be a ring. The **opposite ring** R^{op} has the same underlying abelian group as R, but left multiplication by a in R^{op} is defined as right multiplication by a in R, i.e.*

$$a \cdot_{\mathrm{op}} b = ba.$$

It is clear that $(R^{\mathrm{op}})^{\mathrm{op}} = R$.

Example 1.4: Fields and skew fields are rings. Recall that a skew field (also known as a division ring) is a ring where every nonzero element is invertible.

Example 1.5: Let R be a ring. Then the set of $n \times n$ matrices with entries in R, with matrix addition and multiplication, is also a ring, denoted $\mathrm{Mat}_n(R)$.

Definition 1.6: *A **(left) module** M over a ring R is an abelian group equipped with a ring homomorphism $R \to \mathrm{End}(M)$. Equivalently, we have a bilinear map $R \times M \to M$ satisfying $r_1(r_2(m)) = (r_1 r_2)(m)$.*
*A **submodule** N of M is a subgroup of M closed under the action of R. Given such $N \subset M$, we can also equip M/N with the structure of an R-module.*

Example 1.7: If R is a field, then R-modules are vector spaces.

Definition 1.8: *A **bimodule** over R is a module with compatible R-module and R^{op}-module structures, i.e. the actions of R and R^{op} commute.*

Example 1.9: R is an R-bimodule; the R-module structure is left multiplication and the R^{op}-module structure is right multiplication, and the associativity of multiplication in R implies that these are compatible.

Definition 1.10: *A **left ideal** of R is an R-submodule of R. A **right ideal** of R is an R^{op}-submodule of R (treated as an R^{op}-module). A **two-sided ideal** of R is a subbimodule of R.*

Remark 1.11: If I is a left ideal, as described in Definition 1.6, R/I is an R-module, and likewise, if I is a right ideal, R/I is an R^{op}-module. If I is a two-sided ideal, then the multiplication of elements in R/I is well-defined, and R/I is a ring.

Definition 1.12: *An R-module M is **free** if it is isomorphic to $\bigoplus_{i\in I} R$, where I is some (possibly infinite) index set. If $M \cong R^n$, we say that M has **rank** n. Note that rank is not well-defined in general!*

Example 1.13: Every module over a skew field is free. (See linear algebra.)

Remark 1.14: Remember that in the finite case, direct products and direct sums are the same, but in the infinite case, they are not. In an infinite direct sum, all but finitely many elements must be 0.

1.2 Invariant Basis Number Property

Definition 1.15: *A ring R has the **invariant basis number (IBN) property** if free modules of different ranks are not isomorphic. That is, rank is well-defined.*

Example 1.16: Linear algebra tells us that modules over a skew field satisfy the IBN.

Lemma 1.17: *If $\varphi\colon R \longrightarrow S$ is a ring homomorphism and S satisfies IBN, then so does R.*

Proof. To simplify the discussion, let's focus on finite rank modules. Then $\text{Hom}_R(R^n, R^m) = \text{Mat}_{n,m}(R^{\text{op}})$ ($\text{End}_R(R) = R^{\text{op}}$ because any map $R \longrightarrow R$ commutes with left multiplication, hence is defined by its value at 1, and this can be extended to R^n). If R doesn't satisfy IBN, there exist non-square matrices $A \in \text{Mat}_{n,m}(R^{\text{op}})$, $B \in \text{Mat}_{m,n}(R^{\text{op}})$ so that $AB = 1_m, BA = 1_n$. But applying φ, we then see that $\varphi(A), \varphi(B)$ give an isomorphism between S^n and S^m, contradiction. □

Corollary 1.18: *Any ring admitting a homomorphism into a skew field satisfies IBN.*

Example 1.19: By Zorn's lemma, every commutative ring R has a maximal ideal $\mathfrak{m}$. Then $R \twoheadrightarrow R/\mathfrak{m}$, which is a field, so R has the IBN.

Example 1.20: We will see later that every left Noetherian ring maps to $\text{Mat}_n(D)$ for some n, D a skew field, so it satisfies IBN.

Example 1.21: Let $V = \mathbb{C}^\infty = \bigoplus_{i=1}^{\infty} \mathbb{C}$. Then $R := \text{End}(V)$ doesn't satisfy IBN. Choose subspaces V_1, V_2 such that $V = V_1 \oplus V_2$ and $V \cong V_1, V_2$. Then consider the ideals $I_i := \{r \mid r|_{V_i} = 0\}$. $R = I_1 \oplus I_2$, but also $R \cong I_1, I_2$.

Corollary 1.22: $R = \text{End}(\mathbb{C}^\infty)$ *does not admit a homomorphism into a skew field.*

1.3 Simple modules, Schur Lemma

Theorem 1.23: *Suppose that every R-module is free. Then R is a skew field.*

To prove this, we will use the Schur Lemma about simple modules.

Definition 1.24: *A module M is **simple** or **irreducible** if $M \neq 0$ and it has no nontrivial proper submodules.*

Example 1.25: R is simple over itself iff R is a skew field. (If R is simple over itself, then R has no nontrivial ideals, so every nonzero element must be invertible.)

Lemma 1.26 (Schur): *If M is simple, then $\mathrm{End}_R(M)$ is a division ring.*

Proof. Suppose $\varphi \colon M \to M$ is nonzero. Then $\ker\varphi \neq M$, but M is simple, so $\ker\varphi = 0$. Hence φ is injective. Likewise, $\operatorname{im}\varphi \neq 0$, so $\operatorname{im}\varphi = M$ and φ is surjective. Thus φ is invertible. □

Corollary 1.27: *Any nonzero map of simple modules is an isomorphism. In particular, if M, N are non-isomorphic simple modules, $\mathrm{Hom}(M, N) = 0$.*

Lemma 1.28:

a) *Every nonzero ring has a simple module.*
b) *Every proper left ideal in a nonzero ring is contained in a maximal ideal.*
c) *A proper submodule N in a module M is maximal iff M/N is simple.*

Proof. a) will follow from b) and c) because maximal left ideals of R are maximal R-submodules of R. c) is true because the submodules of M/N are in bijection with the submodules of M containing N.
b) follows from Zorn's Lemma. Its conditions are satisfied because for a nested collection $M_0 \subset M_1 \subset \cdots \subset$ of proper submodules in a finitely generated M, $\bigcup M_i = M$ iff some $M_i = M$. □

Remark 1.29: Part b) is also true for finitely generated modules. If M is not finitely generated, b) may not be true. For example, let $R = \mathbb{Z}, M = \mathbb{Q}$. Then M has no maximal proper submodule because you can find a nested collection of submodules of M whose union is also M.

Corollary 1.30: *Every finitely generated module has an irreducible quotient.*

Proof (of Theorem 1.23). Let L be a simple R-module (that exists by Lemma 1.28 *a*)). It doesn't contain any submodule isomorphic to R^2 because every nonzero element of L generates L. So if L is free, it must be isomorphic to R. But then $\mathrm{End}_R(L) \cong \mathrm{End}_R(R) = R^{\mathrm{op}}$, and $\mathrm{End}_R(L)$ is a skew field by Lemma 1.26. □

1.4 Semisimple modules

Definition 1.31: *A module is **semisimple** if it's isomorphic to a direct sum of simple modules.*

Example 1.32: $\mathbb{C}[t]/(t^2)$ is not semisimple as a module over itself. However, we do have an exact sequence of $\mathbb{C}[t]/(t^2)$-modules:

$$0 \to \mathbb{C}[t]/(t) \to \mathbb{C}[t]/(t^2) \to \mathbb{C}[t]/(t) \to 0.$$

Lemma 1.33: *Let $M = \bigoplus_{i\in I} M_i$ be a semisimple module, M_i are simple modules. Then any submodule $N \subset M$ has a direct complement of the form $\bigoplus_{i\in J} M_i$ for some $J \subset I$.*

Proof. Define $S_J := \bigoplus_{i\in J} M_i$ for $J \subset I$. Consider $J \subset I$ such that $S_J \cap N = 0$; check that the union of a nested collection of these J is a subset J' with $S_{J'} \cap N = 0$. Then there exists a maximal such J. $S_J \cap N = 0$ by construction, and $S_J + N = M$. If not, there exists $M_i \not\subset S_J + N$, and we could then replace J with $J \cup \{i\}$, contradiction. □

Theorem 1.34: *Every R-module is semisimple iff $R = \prod_{i=1}^{n} \mathrm{Mat}_{n_i}(D_i)$ where the D_i are skew fields.*

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