Lecture 2: Semisimple Modules, Socles, Artinian Rings, Wedderburn's Theorem

2.1 More on semisimple modules

Example 2.1: Let D be a skew field. Then D^n is a simple module over $\mathrm{Mat}_n(D)$: given any nonzero vector $v \in D^n$, there's a change of basis matrix M such that $Mv = (1, 0, \dots, 0)$, and we can then use permutation matrices to get all the other basis vectors. Therefore, $\mathrm{Mat}_n(D)(v) = D^n$.

Corollary 2.2: *Subquotients and sums of semisimple modules are semisimple.*

Proof. First, we show that submodules of semisimple modules are semisimple. Let $M \cong \bigoplus_{i \in I} L_i$ and $N \subset M$ a submodule. Then by Lemma 1.33, $N \oplus \bigoplus_{i \in J} L_i \cong M$. Therefore, the composition

$$N \hookrightarrow N \oplus \bigoplus_{i \in J} L_i \cong M \twoheadrightarrow \bigoplus_{i \in J \setminus I} L_i$$

is an isomorphism and N is semisimple.
Then quotients of semisimple M are of the form M/N for N a submodule, so by the above $M/N \cong \bigoplus_{i \in J} L_i$ and is semisimple.
Finally, $\sum M_i$ is semisimple because there is a surjection $\bigoplus M_i \twoheadrightarrow \sum M_i$, so $\sum M_i$ is a quotient of the semisimple module $\bigoplus M_i$. □

Example 2.3: $\mathrm{Mat}_n(D)$ is semisimple over itself. It can be decomposed as $\bigoplus_{i=1}^n \mathrm{Mat}_n(D)(e_i)$ where e_i are the standard basis vectors: each summand is matrices that have zeroes everywhere except the ith column. Therefore, $\mathrm{Mat}_n(D)(e_i) \cong D^n$; combined with Example 2.1, $\mathrm{Mat}_n(D)$ is then semisimple.

2.2 Socles

Definition 2.4: *The **socle** of a module M, denoted $\mathrm{Soc}(M)$, is the sum of all semisimple (or simple) submodules of M. Equivalently, it is the maximal semisimple submodule of M.*

Example 2.5: Let $M = \mathbb{C}[t]$ as a $\mathbb{C}[t]$-module. Then $\mathrm{Soc}(M) = 0$. Submodules of M are ideals in $\mathbb{C}[t]$, and an ideal is simple iff it contains no other ideals. But if $I \neq 0$, $tI \subsetneq I$, so (0) is the only simple submodule of M.

Example 2.6: Let $M = \mathbb{C}[t]/t^n$ as a $\mathbb{C}[t]$-module. Then $\mathrm{Soc}(M) = t^{n-1}M$ and is one-dimensional. The submodules of M are all of the form $t^m M$, so they are simple iff $m = n-1$; otherwise, $t(t^m M) \subsetneq t^m M$. Hence the only simple submodule of M is $t^{n-1}M$.

Example 2.7: Let G be a finite p-group and k be a field of characteristic p. Let $M = k[G]$ as a $k[G]$-module. Then $\mathrm{Soc}(M) = k$. To see that, we will show that the only simple G-module is k. We will induct on the order of G. Our base case is $G = \mathbb{Z}/p\mathbb{Z}$. Let V be a simple G-module. Because $(\sigma - 1)^p = 0$ for all $\sigma \in G$, $\ker(\sigma - 1) \neq 0 \Rightarrow \ker(\sigma - 1) = V$. So $\sigma = 1$ and V must be the trivial representation.
Now suppose G is an arbitrary p-group and V an irreducible G-module. Then G has a nontrivial center (can be shown by using the class equation), and the center must contain $\mathbb{Z}/p\mathbb{Z}$. In particular $\mathbb{Z}/p\mathbb{Z}$ is a normal subgroup of G, so $V^{\mathbb{Z}/p\mathbb{Z}}$ is a nonzero $G/(\mathbb{Z}/p\mathbb{Z})$-representation. By induction, it contains a copy of the trivial representation, and so V has a G-invariant vector. So $0 \neq V^G \subset V$ and V must be trivial.

2.3 Isotypic components

For a semisimple module $M \cong \bigoplus_i L_i$, the direct sum decomposition is not canonical; for example, vector spaces have many different bases. But we see that the multiplicity of each L_i is fixed: the number of summands L_i isomorphic to L is $\dim_D(\mathrm{Hom}(L, M)), D = \mathrm{End}(L)^{\mathrm{op}}$. Moreover, the sum of such L_i is well-defined because it is generated by the images of all maps $L \to M$ (in fact, all embeddings $L \hookrightarrow M$).

Definition 2.8: *Using the above notation, **the L-isotypic component** of M is the sum of the images of all embeddings $L \hookrightarrow M$. Equivalently, if $M \cong \bigoplus L_i$, it is $\bigoplus_{L_i \cong L} L_i$.*

Proposition 2.9: *M is semisimple iff any short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ splits.*

Proof. If M is semisimple, Lemma 1.33 and Corollary 2.2 imply that every short exact sequence of the above form splits.
So suppose that every short exact sequence of the above form splits. Consider the short exact sequence $0 \to \mathrm{Soc}(M) \to M \to N \to 0$; thus we can write $M = \mathrm{Soc}(M) \oplus N$ and the module N has no simple submodules. Notice that any submodule of $N' \subset N$ is a summand of N: consider the complement of $N' + \mathrm{Soc}(M)$ in M and project down to N. Now take some $a \neq 0, a \in N$ and let $N' := Ra \subset N$. By Corollary 1.30, N' has a simple quotient, say L, and by the same argument L must be a summand of N. But N has no simple submodules, a contradiction. □

2.4 Classification of semisimple rings

Theorem 2.10: *Every R-module is semisimple iff R is semisimple over itself iff $R = \prod_{i=1}^n \mathrm{Mat}_{n_i}(D_i)$ where the D_i are skew fields. (This is an augmented version of Theorem 1.34.)*

Proof. The first equivalence comes from the fact that every R-module is a quotient of a free module, so if R is semisimple, so is R^I, and so are any quotients of R^I (see Corollary 2.2).
If R is a finite product of matrix rings, Example 2.3 implies that R is semisimple over itself.
To show the last implication, assume R is semisimple over itself and write $R = \bigoplus L_i$. This sum is finite because R is cyclic (it is generated by 1), so if the sum were over an index set $i \in I$, we could write $1 = \sum_{i \in J \subset I} l_i$ where $|J| < \infty$ and $l_i \in L_i$, so $R = \bigoplus_{i \in J} L_i$. (The same argument would work for any finitely generated module.) Anyway, write

R as the sum of its isotypic components, say

$$\bigoplus_{j\in J} L_j^{d_j}, L_j \neq L_{j'} \Leftrightarrow j \neq j'.$$

We know that

$$R^{\mathrm{op}} = \mathrm{End}_R(R) = \mathrm{End}_R\left(\bigoplus_{j\in J} L_j^{d_j}\right) = \prod_{j\in J} \mathrm{Mat}_{d_j}(\mathrm{End}_R(L_j))$$

and if we let $D_j = (\mathrm{End}_R(L_j))^{\mathrm{op}}$, we get an isomorphism

$$R \cong \prod_{j\in J} \mathrm{Mat}_{d_j}(D_j).$$

□

Remark 2.11: It would seem natural to call rings R semisimple over themselves semisimple. However, there is a separate notion of a simple ring, and not all simple rings are semisimple over themselves (see Example 2.13 below).

2.5 Simple rings and Wedderburn's Theorem

Definition 2.12: *A ring R is* ***simple*** *if R has no 2-sided ideals except for 0 and R.*

Example 2.13: $R = \mathbb{C}\langle x, \partial_x\rangle$ is simple but not semisimple. To see that R is not semisimple, consider $R/R(x\partial_x)$. This module has a surjection to $R/R(\partial_x)$ that does not split (exercise).

Definition 2.14: *A ring R is* ***left (resp. right) Noetherian*** *if every ascending chain of left (resp. right) ideals of R stabilizes (called the ascending chain condition). Equivalently, every left (resp. right) ideal is finitely generated.*

Definition 2.15: *A ring R is* ***left (resp. right) Artinian*** *if every descending chain of left (resp. right) ideals of R stabilizes (the descending chain condition).*

Warning : Being left Artinian/Noetherian is not equivalent to being right Artinian/Noetherian!

Theorem 2.16 (Wedderburn): *Let R be a ring. TFAE:*
 a) *R is simple and (either left or right) Artinian,*
 b) *every R-module is semisimple and R has a unique simple module up to isomorphism,*
 c) *$R \cong \mathrm{Mat}_n(D)$ where D is a skew field.*

Proof. The equivalence of b) and c) follows from Theorem 2.10: if R is a finite product of matrix rings over skew fields, check that $\mathrm{Mat}_n(D)$ is simple over itself, and so R has a unique simple module iff the product only contains one matrix ring. This also shows that c) implies a).
So suppose that R is left Artinian and simple. Then R has a minimal left ideal (because any descending chain of left ideals will stabilize), call it L. Notice that $LR = \sum_{x\in R} Lx$ is a nonzero two-sided ideal, hence all of R, and $R = LR$. So R as a left R-module is a quotient of $\bigoplus_{x\in R} L$, and R is semisimple over itself. Thus a) implies b) by use Theorem 2.10. □