Lecture 3: Isotypic Decomposition, Density Theorem, Noetherian and Artinian Properties, Jacobson Radical

3.1 $k[G]$-modules

Example 3.1: Let G be a finite group and k a field of characteristic not dividing $|G|$ (for simplicity, let's say $\operatorname{char} k = 0$, but the result holds in general). Then all $k[G]$-modules are semisimple.
We will show that every short exact sequence $0 \to N \to M \to L \to 0$ splits. WLOG, we can assume that L is finite-dimensional. Tensoring with L^* and using the fact that $\operatorname{Hom}_G(V, W) = (V^* \otimes W)^G$ when V is finite-dimensional, it suffices to show that for $M \twoheadrightarrow L$, the restriction to $M^G \to L^G$ is also onto. But this is true because given $v \in L^G$, choose any preimage of v in M, say $\tilde{v}$, and consider $\frac{1}{|G|} \sum g(\tilde{v})$, which lies in M^G and maps to v.

Corollary 3.2: *Suppose that k is algebraically closed and $|G|$ does not divide $\operatorname{char} k$. Then $|G| = \sum (\dim \rho_i)^2$ where the ρ_i are the isomorphism classes of simple $k[G]$-modules.*

Proof. The only finite skew field extensions of k are trivial if k is algebraically closed. Hence, by Theorem 2.10 $k[G]$ semisimple means it can be written as $\prod_{i=1}^n \operatorname{Mat}_{d_i}(k)$, and the simple $k[G]$-modules are exactly k^{d_i}, while the dimension of $k[G]$ over k is $\sum d_i^2$. □

3.2 Density Theorem

Theorem 3.3 (Density Theorem): *Let L be a simple R-module and $D = \operatorname{End}_R(L)$. Then given any finite set $x_1, \ldots, x_n, y_1, \ldots, y_n \in L$ with the x_i linearly independent over D, there exists $r \in R$ such that $r(x_i) = y_i$.*

Proof. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$. We want to show that the map $R \to L^n$ taking $r \mapsto r\boldsymbol{x}$ is onto. Suppose that $R\boldsymbol{x} \subset L^n$ is a proper submodule, say N. Since L^n is semisimple, we can then decompose $L^n = N \oplus S$, $S \neq 0$. Therefore, $D^n = \operatorname{Hom}_R(L, L^n) = \operatorname{Hom}_R(L, N) \oplus \operatorname{Hom}_R(L, S)$. Therefore, there exists some $(d_1, \ldots, d_n) \in D^n$ annihilating the proper subspace $\operatorname{Hom}(L, N)$ (acting via the dot product), so the x_i are linearly dependent, a contradiction. □

Remark 3.4: The submodules in an isotypic component $L^n \subset M$ are in bijection with vector subspaces in $D^n, D = \operatorname{End}(L)$. The correspondence sends $N \subset L^n$ to $\operatorname{Hom}(L, N) \subset \operatorname{Hom}(L, L^n) = D^n$ (exercise).

Corollary 3.5: *If L is finite-dimensional simple over $D := \operatorname{End}_R(L)$, then there is a surjection $R \twoheadrightarrow \operatorname{End}_D(L) \cong \operatorname{Mat}_n(D), n = \dim_D(L)$.*

Example 3.6: This is not true if M is infinite-dimensional over D. For example, let $R = \operatorname{End}(\mathbb{C}^\infty)$ and $M = \mathbb{C}^\infty$. Then $D = \operatorname{End}_R(M) = \mathbb{C}$ but there is no surjection $R \to \operatorname{End}_D(M)$ (see Corollary 1.22).

3.3 Noetherian and Artinian modules

Definition 3.7: *A module is **Noetherian (resp. Artinian)** if every ascending (resp. descending) chain of submodules stabilizes.*

Remark 3.8: We'll see that every Artinian ring is also Noetherian, but this is not true for modules.

Example 3.9: Let $R = \mathbb{Z}$. Then $M = \mathbb{Z}$ is a Noetherian module, but it is not Artinian because $(p) \supset (p^2) \supset (p^3) \supset \cdots$ is an infinite descending chain of submodules. Meanwhile, $N = \mathbb{Q}/\mathbb{Z}$ is Artinian, but it is not Noetherian, because $\frac{1}{p}N \subset \frac{1}{p^2}N \subset \cdots$.

Proposition 3.10: *A module is Noetherian iff every submodule is finitely generated.*

Proof. Let M be an R-module. If every $N \subset M$ is finitely generated, suppose we had an ascending chain of submodules $M_1 \subset M_2 \subset \cdots \subset$ and consider $N = \bigcup M_i$. Because N is finitely generated, say with generators $x_1, \dots, x_d$, there exists some i with $M_i \supset \{x_1, \dots, x_d\}$, and the ascending chain stabilizes at M_i.
Now suppose that M is Noetherian and $N \subset M$ is a submodule. Obtain a list of generators $x_i \in N$ by taking $x_1 \neq 0$ and x_i any element not in $N_{i-1} := \langle x_1, \dots, x_{i-1} \rangle$. The ascending chain $N_1 \subset N_2 \subset \cdots$ must stabilize eventually, say at N_d, and $x_1, \dots, x_d$ then generate N. □

Proposition 3.11: *If $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence and M_1, M_2 are Noetherian (resp. Artinian), then M is also Noetherian (resp. Artinian).*

Proof. Clear. □

3.4 Composition Series

Definition 3.12: *A **composition series** of a module M is a filtration $M_0 = 0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$ where M_i/M_{i-1} is simple for all i. That is, the filtration has simple associated graded subquotients. If M has a composition series, we say that it is of **finite length** and say that M has **length** n.*

Lemma 3.13: *A module M has finite length iff M is both Noetherian and Artinian.*

Proof. First, suppose M has a composition series. Then induct on the length of M. If M has length 1, it's simple, and therefore both Noetherian and Artinian. If M has length n, then $0 \to M_{n-1} \to M \to L \to 0$ and M_{n-1}, L are Noetherian and Artinian by induction, so M also is.
Now suppose M is both Noetherian and Artinian. Because M is Artinian, by Zorn's Lemma any nonempty collection of submodules has a minimal element. So let $M_1 \subset M$ be a minimal nonzero submodule; it must be a simple submodule. Now inductively define M_{i+1} to be the minimal submodule properly containing M_i; this will exist unless $M_i = M$, and M_{i+1}/M_i will be simple. This chain of submodules will terminate because M is Noetherian, so $M_n = M$ for some n and we have constructed a composition series for M. □

Definition 3.14: *Let $M_1 \subset \cdots \subset M_n = M$ be a composition series for M. The **associated graded** of the composition series is*

$$\mathrm{gr}(M) := \bigoplus_{i=1}^{n} M_i/M_{i-1}.$$

Theorem 3.15 (Jordan-Hölder): *Given two composition series M_i, M_i' of M, $\mathrm{gr}(M) = \mathrm{gr}'(M)$. Equivalently, the number of irreducible subquotients isomorphic to a given simple module L is independent of the choice of filtration.*

Proof. Induct on the length of M_i. If M_i has length 1, M is simple and both filtrations contain only M with multiplicity 1. If not, consider the smallest j such that $L = M_1 \subset M_j'$. Since $L \not\subset M_{j-1}'$, there is a nonzero map $L \to M_j'/M_{j-1}' = \mathrm{gr}_j'(M)$, and a nonzero map between simples is an isomorphism. Hence $\mathrm{gr}_j'(M) \cong L$.
Therefore, M/M_1 has two filtrations: one given by $\bar{M}_i = M_{i+1}/M_1$ and one defined by $\bar{M}_i'$ is the image of M_i' when $i < j$ and M_{i+1}'/M_1 when $i \geqslant j$. We know that we get $\overline{\mathrm{gr}}(M) = \overline{\mathrm{gr}}'(M)$ from removing one copy of L from $\mathrm{gr}(M)$ and $\mathrm{gr}'(M)$, so by induction, $\mathrm{gr}(M) = \mathrm{gr}'(M)$. □

Remark 3.16: Inspecting the proof of Theorem 3.15, we see that a stronger version of it holds. This stronger version claims that for two composition series $0 \subset M_1 \subset \ldots \subset M_a = M$, $0 \subset M_1' \subset \ldots \subset M_b' = M$ of M there exists a *canonical* bijection $\sigma\colon \{1, \dots, a\} \xrightarrow{\sim} \{1, \dots, b\}$ and a *canonical* isomorphism $M_i/M_{i-1} \xrightarrow{\sim} M_{\sigma(i)}'/M_{\sigma(i)-1}'$. This version of the theorem is interesting already for $R = k$ (so M is a finite-dimensional vector space): in this case, composition series of M are flags of subspaces in M, and σ describes a "relative position" of these two flags with respect to each other.

Definition 3.17: *Let $\mathcal{M}$ be a collection of R-modules closed under subquotients. The **Grothendieck group** $K(\mathcal{M})$ is the free abelian group generated by $[M]$, $M \in \mathcal{M}$, subject to the relations $[M] = [M_1] + [M_2]$ when there is a SES $0 \to M_1 \to M \to M_2 \to 0$.*

Remark 3.18: For A an abelian group, any function $\mathcal{M} \to A$ additive on subquotients then induces a map $K(\mathcal{M}) \to A$. For example, if $R = D$ and $\mathcal{M}$ consists of the finite-dimensional vector spaces, dimension is such a function.

Corollary 3.19: *Let $\mathcal{M}$ be the modules of finite length over R. Then $K(\mathcal{M})$ is freely generated by $[L]$ for (isomorphism classes of) irreducible modules L.*

Proof. The existence of a composition series for each $M \in \mathcal{M}$ means that the $[L]$ generate $K(\mathcal{M})$. To see that the $[L]$ have no relations, notice that Jordan-Hölder implies that there's a well-defined homomorphism $K(\mathcal{M}) \to \mathbb{Z}$ sending $[M]$ to the multiplicity of L in the Jordan-Hölder series of M. Thus every $[M]$ has a unique decomposition into the $[L]$. □

3.5 Jacobson Radical

Definition 3.20: *The **Jacobson radical** $J = J(R)$ of a ring R is the intersection of the annihilators of all simple R-modules.*

The Jacobson radical has many characterizations.

Lemma 3.21: *For $a \in R$ TFAE:*

a) $a \in \operatorname{Ann}(L)$ *for all simple R-modules L (i.e., $a \in J(R)$),*
b) $a \in I$ *for all maximal left ideals I,*
c) $1 - xa$ *has a left inverse for all x,*
d) $1 - xay$ *has an inverse for all x, y,*
e) $1 - ax$ *has a right inverse for all x,*
f) $a \in I$ *for all maximal right ideals I,*
g) $a \in \operatorname{Ann}(L)$ *for all simple R^{op}-modules L (i.e., $a \in J(R^{\mathrm{op}})$).*