Lecture 4: Socle and Cosocle Filtrations, Jacobson Radical, Krull-Schmidt

4.1 Socle and cosocle filtrations

Definition 4.1: *The **socle filtration** of a module $M$is defined inductively as follows: M_1 is the socle of M (see Definition 2.4) and M_i is the preimage of the socle of M/M_{i-1} in M.*

Remark 4.2: The socle filtration can be generalized to transfinite numbers (e.g. ordinals), in which case it is called the **Loewy filtration**, but we won't talk about it.

Definition 4.3: *The **cosocle filtration** of an Artinian module M is also defined inductively: M^1 is the kernel of the map from M to its maximal semisimple quotient (called the **cosocle**), M^2 is the kernel of the map from M^1 to its cosocle, and so on.*

Remark 4.4: If M is Artinian, then the cosocle filtration always exists, but this is not true in general because M may not necessarily have a maximal semisimple quotient. One could consider all possible simple quotients $M \twoheadrightarrow L_i$ and get a map $M \longrightarrow \prod L_i$, but this infinite product need not be semisimple. For example, this occurs when $R = \mathbb{Z}$; then $\prod_p \mathbb{Z}/p\mathbb{Z}$ is not semisimple.
But if M is Artinian, we know the intersection of the kernels of all maps $M \twoheadrightarrow L_i$ is equal to the intersection of the kernels of finitely many such maps: we can order the kernels of all maps $M \longrightarrow \prod_{i=1}^n L_i$ to create a decreasing sequence of submodules, which must stabilize. Hence, there exists a maximal quotient corresponding to the stabilized kernel, $M \longrightarrow \prod_{i=1}^n L_i$. By definition, any map $M \longrightarrow N$ where N is semisimple factors through this image, so $\prod_{i=1}^n L_i$ is the maximal semisimple quotient.

Example 4.5: Let $R = \mathbb{C}[t]$ and suppose that M is a finite-dimensional R-module where t acts nilpotently. Then $M_i = \ker(t^i)$ and $M^i = \mathrm{im}(t^i)$.

4.2 Jacobson radical cont.

Proof (of Lemma 3.21). a) implies b): for any $\mathfrak{m} \subset R$, $R/\mathfrak{m}$ is simple, so a annihilates $R/\mathfrak{m} \Rightarrow a \in \mathfrak{m}$.
b) implies a): the annihilator of every simple module is a proper ideal in R, thus contained in some maximal left ideal.
c) implies b): if there exists a maximal ideal $\mathfrak{m}$ with $a \notin \mathfrak{m}$, then there exists x such that $xa \equiv 1 (\mathrm{mod}\, \mathfrak{m})$. Hence $1 - xa$ is not invertible.
b) implies c): first note that $t \in R$ is left invertible iff $Rt = R$ iff t does not belong to a proper left ideal. By Zorn's Lemma, this is equivalent to $t \notin \mathfrak{m}$ for some maximal left ideal. So if $a \in \mathfrak{m}$ for all maximal $\mathfrak{m}$, $1 - xa \notin \mathfrak{m}$ and $1 - xa$ is left invertible.
d) implies c) follows from setting $y = 1$.
c) implies d): the set of all a satisfying a), b), c) forms a 2-sided ideal by a). So xay also lies in this ideal and $1 - xay$ has a left inverse by c); say it is $1 - b$. Then $(1 - b)(1 - xay) = 1$, and so b also lies in the two-sided ideal. By c), $1 - b$ then has a left inverse, which implies that $1 - xay$ is invertible.
Since d) is left-right symmetric, e), f), and g) follow. □

Remark 4.6: If a is nilpotent with $a^n = 0$, then $1 - a$ is invertible with inverse $1 + a + \cdots + a^{n-1}$. Hence, if xay is nilpotent for all x, y, then $a \in J$.

Example 4.7: Let $R = \mathbb{C}[x_1, \ldots, x_n]/I$. The Jacobson radical of R is $\sqrt{I}/I$, which follows from Hilbert's Nullstellensatz.

Example 4.8: If R is a commutative local ring, then $J(R) = \mathfrak{m}$, the unique maximal ideal.

Example 4.9: If $R \subset \mathrm{Mat}_n(k)$ is the subalgebra of upper triangular matrices, then $J(R)$ is the *strictly* upper triangular matrices (zeroes on the diagonal).

4.3 Local rings and indecomposable modules

Definition 4.10: *A ring R is **local** if all non-invertible elements form an ideal, in which case said ideal is $J(R)$. If R is local, $R/J(R)$ is a skew field.*

Definition 4.11: *A module M is **indecomposable** if it cannot be decomposed as a direct sum of nonzero submodules $M_1 \oplus M_2$.*

Example 4.12: Let $R = \mathbb{C}[t]$, $M = \mathbb{C}^n$, and t acts by some matrix A. Then M is indecomposable iff A has only one Jordan block.

Remark 4.13: M is indecomposable iff $\mathrm{End}_R(M)$ has no nontrivial idempotents, i.e. elements e such that $e^2 = e$. If $e \in \mathrm{End}_R(M)$, then we could write $M = Me \oplus M(1-e)$: $\ker e = \mathrm{im}(1-e)$ because $(1-e)^2 = 1-e$, so $em = 0 \Leftrightarrow (1-e)m = m \Leftrightarrow (1-e)n = m$ for some n.
Conversely, given a decomposition $M = M_1 \oplus M_2$, we could set $e = \pi_{M_1} : M \twoheadrightarrow M_1$.

Remark 4.14: If we took an idempotent of R instead of $\mathrm{End}_R(M)$, we would still get a splitting $M = eM \oplus (1-e)M$, but this would only be a direct sum of abelian groups, not of R-modules.

Proposition 4.15: *If M is indecomposable of finite length, then $\mathrm{End}_R(M)$ is local.*

Lemma 4.16: *If M is an indecomposable finite length module, every $a \in \mathrm{End}_R(M)$ is either nilpotent or invertible.*

Proof. For every $a \in \mathrm{End}(M)$, consider the chains $\ker(a) \subset \ker(a^2) \subset \cdots \subset$ and $\mathrm{im}(a) \supset \mathrm{im}(a^2) \supset \cdots \supset$. Because M is both Artinian and Noetherian, these both stabilize. Let $b = a^n$ where n is such that $\ker(a^{n+1}) = \ker(a^n)$ and $\mathrm{im}(a^{n+1}) = \mathrm{im}(a^n)$. Thus $\ker(b^2) = \ker(b)$, $\mathrm{im}(b^2) = \mathrm{im}(b)$. We claim that then $M = \ker(b) \oplus \mathrm{im}(b)$.
For $x \in \mathrm{End}(M)$, since $\mathrm{im}(b^2) = \mathrm{im}(b)$, there exists y such that $b^2 y = bx$. So $x - by \in \ker(b)$ and $x = (x-by)+by \Rightarrow M = \ker(b) + \mathrm{im}(b)$. To see that it's the direct sum, note that $x \in \ker(b) \cap \mathrm{im}(b)$ implies $x = by$ and $bx = b^2 y = 0$, but $\ker(b^2) = \ker(b)$, so $by = 0 \Rightarrow x = 0$. Hence $\ker(b) \cap \mathrm{im}(b) = \{0\}$.
Since M is indecomposable, either $\ker(b) = 0$ and $\mathrm{im}(b) = M$ or $\mathrm{im}(b) = 0$ and $\ker(b) = M$. If $\ker(b) = 0$ and $\mathrm{im}(b) = M$, then $\ker(a) = 0$ and $\mathrm{im}(a) = M$ also and so a is invertible. If $\mathrm{im}(b) = 0$, then $b = 0$, so a is nilpotent. □

Proof (of Proposition 4.15). If $a \in \mathrm{End}_R(M)$ is not invertible, it's nilpotent. Hence $\ker(a) \neq 0$. So xa is also not invertible, hence nilpotent. By the same argument, xay is also not invertible, hence nilpotent. By Remark 4.6, $1 - xay$ is invertible for all x, y and thus $a \in J(R)$. □

4.4 Krull-Schmidt

Theorem 4.17 (Krull-Schmidt):
a) *Every finite length module can be decomposed as a direct sum of indecomposable modules.*
b) *For any two such decompositions, the multisets of isomorphism classes of the indecomposable summands coincide.*

Example 4.18: Let $R = \mathbb{C}[t]$. Then a finite length module is a finite-dimensional vector space and t acts by a matrix. Indecomposable modules correspond to matrices with a single Jordan block, so in this case, Krull-Schmidt is equivalent to saying every matrix has a (essentially unique) Jordan normal form.

Proof (of Theorem 4.17). The proof that such a decomposition exists only requires our module to be either Noetherian or Artinian, but not both. Suppose that M cannot be written as a direct sum of indecomposables. So M is not indecomposable, which means it has a decomposition $M = M_1 \oplus M_2$ but one of M_1, M_2 is not a direct sum of indecomposables, WLOG M_1. Then we can split M_1, and inductively continue the process indefinitely. This gives us both an infinite descending chain of submodules (the submodules we split at every step) and an infinite ascending chain of submodules (the complement of those submodules), one of which stabilizes, a contradiction.
However, uniqueness requires M to be of finite length.
Let P, Q be any two R-modules. Let $S = \mathrm{End}_R(P)^{\mathrm{op}}$. Then $\mathrm{Hom}_R(P, Q)$ is a left S-module and $\mathrm{Hom}_R(Q, P)$ is a right S-module. Even better, we have a pairing

$$\begin{aligned} \mathrm{Hom}_R(P,Q) \times \mathrm{Hom}_R(Q,P) &\to \mathrm{Hom}_R(P,P) = S \\ (f,g) &\mapsto g \circ f. \end{aligned}$$

If P and Q are indecomposable, then S is local with maximal ideal $\mathfrak{m}_S = J(S)$. Then we claim that the image of this pairing lands in $\mathfrak{m}_S$ iff $P \not\cong Q$. Suppose that there exists f, g with $g \circ f$ invertible; then $Q \cong P \oplus \ker(g)$. This contradicts the indecomposability of Q unless $P \cong Q$.
Now consider $\overline{\mathrm{Hom}_R(P,Q)} := \mathrm{Hom}_R(P,Q)/\mathfrak{m}_S \mathrm{Hom}_R(P,Q)$ and likewise define $\overline{\mathrm{Hom}_R(Q,P)}$. Both of these are

modules over the skew field $D_S := S/\mathfrak{m}_S$, i.e. vector spaces, so we get a D-bilinear pairing

$$\overline{\mathrm{Hom}_R(P,Q)} \times \overline{\mathrm{Hom}_R(Q,P)} \longrightarrow D$$

and this pairing is nonzero iff $P \cong Q$.
Moreover, if Q is not indecomposable, but instead a direct sum $Q_1 \oplus Q_2$, then

$$\overline{\mathrm{Hom}_R(P,Q)} = \overline{\mathrm{Hom}_R(P,Q_1)} \oplus \overline{\mathrm{Hom}_R(P,Q_2)}$$

and likewise for $\overline{\mathrm{Hom}_R(Q,P)}$, and these direct sum decompositions are compatible with the pairing.
Therefore, if $M = \bigoplus_{i=1}^n Q_i$ for Q_i indecomposable, we can likewise decompose $\overline{\mathrm{Hom}_R(P,M)}$ and $\overline{\mathrm{Hom}_R(M,P)}$ and deduce that the number of Q_i isomorphic to a given P is the rank of the pairing $\overline{\mathrm{Hom}_R(P,M)} \times \overline{\mathrm{Hom}_R(Q,M)} \longrightarrow D$.
This is independent of the decomposition, so the multiplicities of the isomorphism classes of the indecomposables are unique. □

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