Lecture 5: Jacobson Radical, Primitive and Semi-Primitive Rings

5 February 23 - Jacobson radical, primitive and semi-primitive rings

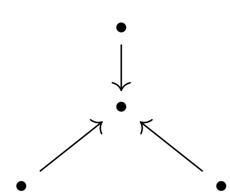
5.1 Interlude on quiver representations

While the indecomposables of $\mathbb{C}[t]$ have a nice classification via Jordan normal form, this is generally a wild problem.

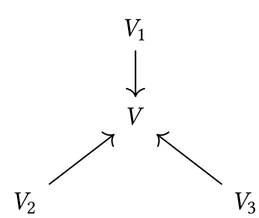
For example, one way we can generalize this is by asking how to parametrize finite sets of subspaces of a vector space V. For example, how can we parametrize triples V_1, V_2, V where $V_1, V_2 \subset V$? Say two triples V_1, V_2, V and V_1', V_2', V' are equivalent when there is an isomorphism $V \cong V'$ that sends $V_i \mapsto V_i'$. This is not so bad – these triples are determined up to equivalence by the integers $\dim V_i$, $\dim V$, and $\dim(V_1 \cap V_2)$.

Another nice example is considering invariants for $V_1, V_2, V_3, V_4 \subset V$ when $V = \mathbb{R}^2$. If we require all 4 to be 1-dimensional subspaces of $\mathbb{R}^2$, we want to parametrize quadruples of lines on the plane. In general position, no two coincide, and an isomorphism will take one configuration to the other when they have the same cross ratio. So in this case, our invariant is a general element of $\mathbb{R}$, not a bunch of integers.

More generally, you could ask how to describe any number of subspaces in a vector space. We can rephrase this question in the language of quiver representations. Recall that a quiver is an oriented graph, and a representation of a quiver is just an assignment of a vector space to each vertex and a map between the corresponding vector spaces for each edge. For example, a representation of the below quiver is 4 vector spaces, one for each vertex, and maps between them.



That is, a representation looks like



and we can define isomorphisms and direct sums of representations, hence speak about indecomposable and simple representations of this quiver.

Representations of a quiver Q are equivalent to modules of its path algebra $A(Q)$. So Krull-Schmidt tells us that the decomposition of a finite-dimensional representation into indecomposables has unique multiplicities. Fact: the above quiver has 12 indecomposables, so there are 12 invariants necessary to describe a representation of this quiver (one for each indecomposable multiplicity). The dimension of V_1, V_2, V_3 is at most 1, while the dimension of V is at most 2, in each indecomposable. In three of these, the maps aren't injective. So quadruples $V_1, V_2, V_3 \subset V$ are parametrized by 9 invariants, and in fact, we can express these explicitly as intersections.

5.2 Primitive and semi-primitive rings

Definition 5.1: *We say a ring R is **semi-primitive** if $J(R) = 0$. Equivalently, $R \hookrightarrow \mathrm{End}(M)$ for some semisimple R-module M. Since $J(R) = J(R^{\mathrm{op}})$, we could also say that $R \hookrightarrow \mathrm{End}(M)$ for a semisimple R^{op}-module M.*

Definition 5.2: *We say a ring R is **(left, right) primitive** if R has a faithful simple (left, right) R-module, that is, $R \hookrightarrow \mathrm{End}(M)$. There exist rings that are left but not right primitive.*

So primitive rings correspond to having a faithful simple module, while semi-primitive rings correspond to having a faithful semisimple module.

Example 5.3: Simple rings are both left and right primitive: every simple module of a simple ring R has to be faithful, because $\ker(R \to \mathrm{End}(L))$ is a 2-sided ideal in R, hence 0.

Example 5.4: Primitive rings need not be simple. For example, $R = \mathrm{End}(\mathbb{C}^\infty)$ is primitive but not simple. It is primitive because $\mathbb{C}^\infty$ is a simple R-module, but R is not simple because operators of finite rank in R form a two-sided ideal.

Example 5.5: Here's a more "real-life" example. consider $R = U(\mathfrak{sl}_2)/(C)$, where C is the Casimir $ef + fe + \frac{h^2}{2}$ (a central element). We claim this is primitive but not simple. First, R can be identified with the ring S of global differential operators on $\mathbb{P}^1$. Verifying this is a hard exercise; here is an outline:
On each copy of $\mathbb{C}$, the differential operators are generated by $x, \frac{\partial}{\partial x}$. To move between copies, note that $\frac{\partial}{\partial x} = -x^{-2}\frac{\partial}{\partial x^{-1}}$. You can show that the global vector fields on $\mathbb{P}^1$ are generated by $\frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x}$, and the Lie algebra they generate (via taking the commutator of vector fields) has the same relations as $\mathfrak{sl}_2$. This gives us a map $U(\mathfrak{sl}_2) \to S$, and you can check that it kills C, so we get a map $R \to S$, and you then show this map is an isomorphism.
(The geometric explanation for this: SL_2 acts on $\mathbb{P}^1$, so we get a map from $\mathfrak{sl}_2$ to the Lie algebra of vector fields on $\mathbb{P}^1$. This has a far-reaching generalization describing differential operators on a flag variety as an appropriate quotient of the universal enveloping algebra modulo an ideal generated by central elements.)
Anyway, to construct a faithful simple R-module, note that the differential operators on $\mathbb{A}^1$ act on $\mathbb{C}[x]$, and this induces an action of differential operators on $\mathbb{P}^1$ on $\mathbb{C}[x]/\mathbb{C}$. Exercise: verify this is in fact a faithful simple R-module.

So now we have several examples where primitive rings need not be simple. However, if we add the condition that our ring must be Artinian (e.g. a finite-dimensional algebra over a field), then every primitive ring is in fact simple.

Proposition 5.6: *A (left or right) Artinian semi-primitive ring has the form $\prod_{i=1}^n \mathrm{Mat}_{n_i}(D_i)$, so (left or right) Artinian primitive rings are of the form $\mathrm{Mat}_n(D)$, hence simple.*

Proof. Suppose that R is Artinian and semi-primitive, i.e. $J(R) = 0$. We can also write $J(R) = \bigcap I_\alpha$ where the intersection is over all maximal left ideals I_α. Because R is Artinian, there exists a finite subset of the I_α such that $\bigcap_{i=1}^n I_i = 0$ (consider the infinite descending chain of ideals $I_1, I_1 \cap I_2, \ldots$. This must stabilize, but also $\bigcap I_\alpha = 0$, so it must stabilize at 0).

Therefore, we have an injection $R \hookrightarrow \bigoplus_{i=1}^n R/I_i$. Because each I_i is maximal, R/I_i is simple, so R is a submodule of a semisimple module, hence is semisimple itself. Then by Theorem 2.10, R is a finite product of matrix algebras. Then the second part follows since the simple representations of R will be $\mathrm{Mat}_{n_i}(D_i)$, and these are not faithful unless $R = \mathrm{Mat}_n(D)$. □

Corollary 5.7: *Suppose R is Artinian. Then M is semisimple iff $J(R)M = 0$. The socle filtration on M has $M_i = \ker J(R)^i$ and the cosocle filtration is $J(R)^i M$.*

Proof. In one direction, if M is semisimple, then by definition $J(R)$ annihilates all simple, hence all semisimple, modules. In the other, suppose that $J(R)$ acts trivially. Then M is a quotient over $R/J(R)$. $J(R/J(R)) = 0$, so this quotient is semi-primitive. It is also Artinian, so M is a module over $\prod_{i=1}^n \mathrm{Mat}_{n_i}(D_i)$. This ring is semisimple, so M is semisimple.

The second statement follows from the first; for example, the socle is the maximal semisimple submodule of M, which must then be the kernel of $J(R)$, and so on. □

Corollary 5.8 (A Version of Nakayama): *Suppose M is a finitely generated R-module such that $J(R)M = M$. Then $M = 0$.*

Proof. If M is nonzero, we know that M has a simple quotient by Corollary 1.30, call it L, and $J(R)L = 0$. Then $J(R)M \neq M$. □