Lecture 6: Artinian Rings are Noetherian, Projective Covers

6.1 The Akizuki-Hopkins-Levitzki Theorem (Artinian rings are Noetherian)

Lemma 6.1: *If R is Artinian, then $J = J(R)$ is a nilpotent ideal, i.e. there exists some $n > 0$ such that $J^n = 0$.*

Proof. Saying that $J^n = 0$ is equivalent to saying that $x_1 x_2 \cdots x_n = 0$ for all $x_i \in J$. Consider the decreasing chain $J \supset J^2 \supset \cdots \supset$, which stabilizes because R is Artinian. So let $I = J^n = J^{n+1}$; then $I = I^2$ also. If $I \neq 0$, there exists a minimal left ideal M such that $IM \neq 0$ (use that R is Artinian). Pick $a \in M$ such that $Ia \neq 0$; then $I(Ia) \neq 0$ and $Ia \subset M$, so $Ia = M$ by minimality of M. Thus, there exists $x \in I$ such that $a = xa$, so $1 - x$ is a zero divisor. But since $x \in J$, $1 - x$ is invertible, contradiction. □

Theorem 6.2 (Akizuki-Hopkins-Levitzki): *If R is (left, right) Artinian, then R has finite length as a (left, right) module over R. In particular, R is Noetherian.*

Proof. We'll show that $M_d := J^d / J^{d+1}$ is a finite length R-module. This module is annihilated by J, so it's semisimple. Recall that semisimple modules are Artinian iff they are Noetherian iff they are a finite sum of irreducibles. But J^d / J^{d+1} is Artinian, so it has a finite length. Then

$$\text{length}(R) = \sum_{i=0}^{n-1} \text{length}(M_n)$$

where the sum is finite because $J^n = 0$, so R has finite length. □

6.2 Projective covers

Definition 6.3: *A module P is* ***projective*** *if $\mathrm{Hom}(P, -)$ is exact (takes short exact sequences to short exact sequences). Equivalently, given a surjection $N \twoheadrightarrow M$, we can lift any map $P \to M$ (non-uniquely) to a map $P \to N$.*

Example 6.4: Free modules are projective. Direct summands of projective modules are also projective, so direct summands of free modules are projective. In fact, the converse is also true, since every projective P has a surjection $R^I \twoheadrightarrow P$, so we can lift $P \cong P$ to $P \longrightarrow R^I$, which gives us a splitting of $R^I = P \oplus Q$.

Corollary 6.5: *Every module is the quotient of a projective module.*

Definition 6.6: *A surjection $\varphi\colon M \twoheadrightarrow N$ is an* ***essential surjection*** *if for all $M' \subsetneq M$, $\varphi|_{M'}$ is not onto. That is, no proper submodule of M surjects onto N.*

Definition 6.7: *A* ***projective cover*** *of a module M is an essential surjection $P \twoheadrightarrow M$ from a projective module P.*

Example 6.8: Let M be a finite length module and M^1 be the first term of the cosocle filtration, so $S := M/M^1 = M/JM$ is the maximal semisimple quotient (see Corollary 5.7). Then $M \twoheadrightarrow S$ is an essential surjection. One way to see this: if $N \subset M$ and $N \twoheadrightarrow S = M/JM$, then $(M/N)/J(M/N) = 0$. So by Nakayama $M/N = 0$. In fact, any essential surjection $M \twoheadrightarrow S$ with S semisimple and M finite length has this form.

Lemma 6.9:

a) *Suppose $p\colon P \twoheadrightarrow M$ is a projective cover and $q\colon Q \twoheadrightarrow M$ is another surjection from a projective Q to M. Then we can write $Q \cong P \oplus Q'$ with $q|_{Q'} = 0$ and $q|_P = p$.*

b) *A projective cover (if it exists) is unique up to isomorphism.*

Proof. b) follows from a), so it suffices to prove a). We can lift q to a map $\tilde{q}\colon Q \longrightarrow P$ with $q\colon Q \xrightarrow{\tilde{q}} P \overset{p}{\twoheadrightarrow} M$. Since p is an essential surjection, Q must be onto (as $\mathrm{Im}(\tilde{q}) \twoheadrightarrow M$). But surjective maps between projective modules split, so we get the desired splitting of Q. □

Proposition 6.10: *Suppose R is Artinian.*

a) *Every irreducible module has a projective cover.*

b) *The isomorphism classes of irreducible modules are in bijection with isomorphism classes of indecomposable projectives. This bijection sends L to its projective cover and a projective module to its cosocle (its maximal semisimple quotient).*

Proof. b) follows from a): let P be an indecomposable projective. Since P is a summand of a free, there is a nonzero map from P to R, hence $P \twoheadrightarrow L$ for some irreducible L. But P_L, the projective cover of L, is a direct summand of P by Lemma 6.9, so $P \cong P_L$.
To prove a), it suffices to find a projective P_L such that $P_L/JP_L \cong L$, where $J = J(R)$, since then $P_L \twoheadrightarrow L$ is an essential surjection (see Example 6.8). We will induct on n such that $J^n = 0$. If $n = 1$, R is semi-primitive, and thus $R \cong \prod \mathrm{Mat}_{n_i}(D_i)$. Here everything is projective, so $L = P_L$. In general, we will use the lifting of idempotents; the below lemma will show that we can lift idempotents from R/I to R when $I^2 = 0$.
Suppose $n > 1$, then R/J is semi-primitive, so there exists an idempotent $\bar{e} \in R/J$ such that $(R/J)\bar{e} \cong L$. Then we can lift idempotents repeatedly along surjections $R/J^{d+1} \twoheadrightarrow R/J^d$ until we get some e in R (use Lemma 6.11 below). Then consider $P_L = Re$. This satisfies $P_L/JP_L = (R/J)\bar{e} \cong L$, and P_L is a summand of R, so we are done. □

Lemma 6.11: *Let S be a ring and $I \subset S$ a 2-sided ideal such that $I^2 = 0$. Then any idempotent $e \in R := S/I$ can be lifted to an idempotent $\bar{e} \in S$.*

Proof. Let e' be any lift of e, not necessarily an idempotent. We can decompose I into the direct sum

$$I = e'Ie' \oplus e'I(1-e') \oplus (1-e')Ie' \oplus (1-e')I(1-e').$$

Note that the decomposition above does not depend on the choice of e' (use that $I^2 = 0$). Notice that $\varepsilon := e'(1-e')$ lies in I (as it's 0 mod I). Moreover, it satisfies $e'\varepsilon(1-e') = (1-e')\varepsilon e' = \varepsilon^2 = 0$ (use that $I^2 = 0$), so in the direct sum decomposition ε has only nonzero first and last components. That is, we can write $\varepsilon = \varepsilon_+ + \varepsilon_-$, where $\varepsilon_+ \in e'Ie'$

and $\varepsilon_- \in (1-e')I(1-e')$. Now we claim that

$$\bar{e} := e' + \varepsilon_+ - \varepsilon_-$$

is an idempotent lifting of e. Indeed we have

$$\bar{e}(1-\bar{e}) = (e' + \varepsilon_+ - \varepsilon_-)(1 - e' - \varepsilon_+ + \varepsilon_-) = \varepsilon - e'\varepsilon_+ - \varepsilon_-(1-e') = \varepsilon - e'\varepsilon - \varepsilon(1-e') = 0.$$

□

Remark 6.12: An alternative approach to the proof of Lemma 6.11: let e' be a lift of e and set $f' := 1 - e'$. We have $1 - e'^2 - f'^2 \in I$ is nilpotent so $e'^2 + f'^2$ is invertible and it is easy to see that $e'' = \frac{e'^2}{e'^2 + f'^2}$ is the desired lift of e (use that $e'^2 f'^2 = 0$).

Remark 6.13: Let P_L be the projective cover of L. Then $\mathrm{Hom}_R(P_L, L') = 0$ if $L' \neq L$, and $\mathrm{Hom}_R(P_L, L)$ is a free module over D_L^{op}, where $D_L := \mathrm{End}_R(L)$.

Corollary 6.14: *Let R be an Artinian ring and write $R/J = \prod \mathrm{Mat}_{n_i}(D_i)$, $D_i = \mathrm{End}(L_i)^{\mathrm{op}}$ where the L_i are the isomorphism classes of simple R-modules and $n_i = \dim_{D_i}(L_i)$. Let P_i be the projective cover of L_i. Then*

$$R \cong \bigoplus_i P_i^{d_i}$$

as a left R-module.

Proof. By Theorem 4.17, $R \cong \bigoplus_i P_i^{m_i}$ for some multiplicities m_i. Then $\mathrm{Hom}_R(R, L_i) \cong \mathrm{Hom}_R(P_i^{m_i}, L_i)$, so $L_i \cong D_i^{m_i}$ and $m_i = d_i$. □

Remark 6.15: Suppose A is a finite-dimensional algebra over an algebraically closed field k. Then $\mathrm{End}_A(L) \cong k$ for all irreducible L. Then we get another proof of Theorem 3.15, as in this case, the multiplicity of L_i in M will be $\dim_k \mathrm{Hom}_A(P_L, M)$.

Corollary 6.16: *Let R be an Artinian ring. Then any finitely generated R-module has a projective cover.*

Proof. Induct on length. Consider $0 \to L \to M \to N \to 0$ where L is simple and suppose we know N has projective cover P_N with $\varphi\colon P_N \twoheadrightarrow N$. If $P_N \twoheadrightarrow M$, then P_N is also the projective cover of M. Otherwise, M must split as $L \oplus \mathrm{Im}(\varphi) = L \oplus N$, so $P_L \oplus P_N$ is a projective cover of M. □

6.3 Preview of Morita theory

If the P_i are the indecomposable projectives of a ring R, how is $S := \mathrm{End}_R\left(\bigoplus_i P_i^{m_i}\right)^{\mathrm{op}}$ related to R? It turns out that when $m_i \geqslant 1$, S is **Morita equivalent** to R, meaning that their module categories are equivalent.

Theorem 6.17: *S is Morita equivalent to R iff $S^{\mathrm{op}} = \mathrm{End}_R(P)$, where P is a finitely generated "projective generator" of R-Mod.*

We will precisely define the projective generator next time, but when R is Artinian, it will be when $m_i \geqslant 1$ as mentioned above.

18.706 Noncommutative Algebra
Spring 2023