Lecture 6: Artinian Rings are Noetherian, Projective Covers

6.1 The Akizuki-Hopkins-Levitzki Theorem (Artinian rings are Noetherian)

Lemma 6.1: If $R$ is Artinian, then $J = J(R)$ is a nilpotent ideal, i.e. there exists some $n > 0$ such that $J^n = 0$.

Proof. Saying that $J^n = 0$ is equivalent to saying that $x_1 x_2 \cdots x_n = 0$ for all $x_i \in J$. Consider the decreasing chain $J \supset J^2 \supset \cdots$, which stabilizes because $R$ is Artinian. So let $I = J^n = J^{n+1}$; then $I = I^2$ also. If $I \neq 0$, there exists a minimal left ideal $M$ such that $IM \neq 0$ (use that $R$ is Artinian). Pick $a \in M$ such that $Ia \neq 0$; then $I(Ia) \neq 0$ and $Ia \subseteq M$, so $Ia = M$ by minimality of $M$. Thus, there exists $x \in I$ such that $a = xa$, so $1 - x$ is a zero divisor. But since $x \in J$, $1 - x$ is invertible, contradiction. □

Theorem 6.2 (Akizuki-Hopkins-Levitzki): If $R$ is (left, right) Artinian, then $R$ has finite length as a (left, right) module over $R$. In particular, $R$ is Noetherian.

Proof. We’ll show that $M_d := J^d/J^{d+1}$ is a finite length $R$-module. This module is annihilated by $J$, so it’s semisimple. Recall that semisimple modules are Artinian iff they are Noetherian iff they are a finite sum of irreducibles. But $J^d/J^{d+1}$ is Artinian, so it has a finite length. Then

$$\text{length}(R) = \sum_{i=0}^{n-1} \text{length}(M_i)$$

where the sum is finite because $J^n = 0$, so $R$ has finite length. □

6.2 Projective covers

Definition 6.3: A module $P$ is projective if $\text{Hom}(P, -)$ is exact (takes short exact sequences to short exact sequences). Equivalently, given a surjection $N \twoheadrightarrow M$, we can lift any map $P \rightarrow M$ (non-uniquely) to a map $P \rightarrow N$. 
Example 6.4: Free modules are projective. Direct summands of projective modules are also projective, so direct summands of free modules are projective. In fact, the converse is also true, since every projective $P$ has a surjection $R^i \to P$, so we can lift $P \cong P$ to $P \to R^i$, which gives us a splitting of $R^i = P \oplus Q$.

Corollary 6.5: Every module is the quotient of a projective module.

Definition 6.6: A surjection $\varphi : M \to N$ is an essential surjection if for all $M' \subseteq M$, $\varphi|_{M'}$ is not onto. That is, no proper submodule of $M$ surjects onto $N$.

Definition 6.7: A projective cover of a module $M$ is an essential surjection $P \to M$ from a projective module $P$.

Example 6.8: Let $M$ be a finite length module and $M^1$ be the first term of the cosocle filtration, so $S := M/M^1 = M/JM$ is the maximal semisimple quotient (see Corollary 5.7). Then $M \to S$ is an essential surjection. One way to see this: if $N \subset M$ and $N \to S = M/JM$, then $(M/N)/(M/N) = 0$. So by Nakayama $M/N = 0$. In fact, any essential surjection $M \to S$ with $S$ semisimple and $M$ finite length has this form.

Lemma 6.9:

a) Suppose $p : P \to M$ is a projective cover and $q : Q \to M$ is another surjection from a projective $Q$ to $M$. Then we can write $Q \cong P \oplus Q'$ with $q|_{Q'} = 0$ and $q|_P = p$.

b) A projective cover (if it exists) is unique up to isomorphism.

Proof. b) follows from a), so it suffices to prove a). We can lift $q$ to a map $\hat{q} : Q \to P$ with $q : Q \cong P \to M$. Since $p$ is an essential surjection, $Q$ must be onto (as $\text{Im}(\hat{q}) \to M$). But surjective maps between projective modules split, so we get the desired splitting of $Q$.  

Proposition 6.10: Suppose $R$ is Artinian.

a) Every irreducible module has a projective cover.

b) The isomorphism classes of irreducible modules are in bijection with isomorphism classes of indecomposable projectives. This bijection sends $L$ to its projective cover and a projective module to its cosocle (its maximal semisimple quotient).

Proof. b) follows from a): let $P$ be an indecomposable projective. Since $P$ is a summand of a free, there is a nonzero map from $P$ to $R$, hence $P \to L$ for some irreducible $L$. But $P_L$, the projective cover of $L$, is a direct summand of $P$ by Lemma 6.9 so $P \cong P_L$.

To prove a), it suffices to find a projective $P_L$ such that $P_L/JP_L \cong L$, where $J = J(R)$, since then $P_L \to L$ is an essential surjection (see Example 6.8). We will induct on $n$ such that $J^n = 0$. If $n = 1$, $R$ is semi-primitive, and thus $R \cong \bigoplus \text{Mat}_{n_i}(D_i)$. Here everything is projective, so $L = P_L$. In general, we will use the lifting of idempotents; the below lemma will show that we can lift idempotents from $R/I$ to $R$ when $I^2 = 0$.

Suppose $n > 1$, then $R/J$ is semi-primitive, so there exists an idempotent $\tilde{e} \in R/J$ such that $(R/J)\tilde{e} \cong L$. Then we can lift idempotents repeatedly along surjections $R/J^d \to R/J^d$ until we get some $e$ in $R$ (use Lemma 6.11 below). Then consider $P_L = Re$. This satisfies $P_L/JP_L = (R/J)\tilde{e} \cong L$, and $P_L$ is a summand of $R$, so we are done.

Lemma 6.11: Let $S$ be a ring and $I \subset S$ a 2-sided ideal such that $I^2 = 0$. Then any idempotent $e \in R := S/I$ can be lifted to an idempotent $\tilde{e} \in S$.

Proof. Let $e'$ be any lift of $e$, not necessarily an idempotent. We can decompose $I$ into the direct sum

$$I = e'1e' \oplus e'I(1-e') \oplus (1-e')le' \oplus (1-e')(1-e').$$

Note that the decomposition above does not depend on the choice of $e'$ (use that $I^2 = 0$). Notice that $\varepsilon := e'(1-e')$ lies in $I$ (as it’s 0 mod $I$). Moreover, it satisfies $e'\varepsilon(1-e') = (1-e')e'e' = e'^2 = 0$ (use that $I^2 = 0$), so in the direct sum decomposition $\varepsilon$ has only nonzero first and last components. That is, we can write $\varepsilon = \varepsilon_+ + \varepsilon_-$, where $\varepsilon_+ \in e'1e'$ and $\varepsilon_- \in e'I(1-e')$. Then $\varepsilon_+ + \varepsilon_- \in I$, so we are done.
and \( \varepsilon_- \in (1 - e')(1 - e') \). Now we claim that

\[
\tilde{e} := e' + e_+ - \varepsilon_-
\]

is an idempotent lifting of \( e \). Indeed we have

\[
\tilde{e}(1 - \tilde{e}) = (e' + e_+ - \varepsilon_-)(1 - e' - e_+ + \varepsilon_-) = \varepsilon - e'e_+ - \varepsilon_-(1 - e') = \varepsilon - e'e - \varepsilon(1 - e') = 0.
\]

Remark 6.12: An alternative approach to the proof of Lemma 6.11 let \( e' \) be a lift of \( e \) and set \( f' := 1 - e' \). We have \( 1 - e'^2 - f'^2 \in \mathfrak{I} \) is nilpotent so \( e'^2 + f'^2 \) is invertible and it is easy to see that \( e'' = e' f'^{-1} e' \) is the desired lift of \( e \) (use that \( e'^2 f'^2 = 0 \)).

Remark 6.13: Let \( P_L \) be the projective cover of \( L \). Then \( \text{Hom}_R(P_L, L') = 0 \) if \( L' \neq L \), and \( \text{Hom}_R(P_L, L) \) is a free module over \( D_L^{\text{op}} \), where \( D_L := \text{End}_R(L) \).

Corollary 6.14: Let \( R \) be an Artinian ring and write \( R/J = \prod \text{Mat}_{n_i}(D_i) \), \( D_i = \text{End}(L_i)^{\text{op}} \) where the \( L_i \) are the isomorphism classes of simple \( R \)-modules and \( n_i = \dim_{D_i}(L_i) \). Let \( P_i \) be the projective cover of \( L_i \). Then

\[
R \cong \bigoplus_i p_i^{d_i}
\]

as a left \( R \)-module.

Proof. By Theorem 4.17, \( R \cong \bigoplus_i p_i^{m_i} \) for some multiplicities \( m_i \). Then \( \text{Hom}_R(R, L_i) \cong \text{Hom}_R(P_i^{m_i}, L_i) \), so \( L_i \cong D_i^{m_i} \) and \( m_i = d_i \). □

Remark 6.15: Suppose \( A \) is a finite-dimensional algebra over an algebraically closed field \( k \). Then \( \text{End}_A(L) \cong k \) for all irreducible \( L \). Then we get another proof of Theorem 3.15, as in this case, the multiplicity of \( L_i \) in \( M \) will be \( \dim_k \text{Hom}_A(P_L, M) \).

Corollary 6.16: Let \( R \) be an Artinian ring. Then any finitely generated \( R \)-module has a projective cover.

Proof. Induct on length. Consider \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) where \( L \) is simple and suppose we know \( N \) has projective cover \( P_N \) with \( \varphi: P_N \rightarrow N \). If \( P_N \rightarrow M \), then \( P_N \) is also the projective cover of \( M \). Otherwise, \( M \) must split as \( L \oplus \text{Im}(\varphi) = L \oplus N \), so \( P_L \oplus P_N \) is a projective cover of \( M \). □

6.3 Preview of Morita theory

If the \( P_i \) are the indecomposable projectives of a ring \( R \), how is \( S := \text{End}_R \left( \bigoplus_i p_i^{m_i} \right)^{\text{op}} \) related to \( R \)? It turns out that when \( m_i \geq 1 \), \( S \) is Morita equivalent to \( R \), meaning that their module categories are equivalent.

Theorem 6.17: \( S \) is Morita equivalent to \( R \) iff \( S^{\text{op}} = \text{End}_R(P) \), where \( P \) is a finitely generated “projective generator” of \( R \)-Mod.

We will precisely define the projective generator next time, but when \( R \) is Artinian, it will be when \( m_i \geq 1 \) as mentioned above.
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