Lecture 7: Categories and Morita Equivalence

Remark 7.1: We can also discuss projective covers of graded modules over graded rings. Let $R = \bigoplus_{n\geqslant 0} R_n$ with R_0 Artinian and let L be an irreducible graded module over R that is concentrated in one degree. WLOG we can assume L is concentrated in degree 0. Then $P = Re_L$ is a graded projective cover of L; $e_L \in R_0$ is the idempotent corresponding to the projective cover of L as an R_0-module.

7.1 Morita equivalence

Definition 7.2: *We say that two rings are **Morita equivalent** if their categories of modules are equivalent.*

(Below, we will recall some facts about categories.)

Theorem 7.3: *A ring S is Morita equivalent to a ring R iff $S = \mathrm{End}_R(P)^{\mathrm{op}}$ where P is a finitely generated projective generator of the category of R-modules.*

Definition 7.4: *A projective module P over a ring R is a **projective generator** if $\mathrm{Hom}(P, M) \neq 0$ for every nonzero R-module M.*

Lemma 7.5: *M is a generator iff R is a direct summand in M^n for some n.*

Proof. If M is a generator, then for every module N, the images of all possible homomorphisms $M \to N$ generate N. This is because if S is the sum of all the images of such maps, then $\mathrm{Hom}(M, S) \to \mathrm{Hom}(M, N)$ is an isomorphism, and since M is a generator, this implies that $S \cong N$.
Now if N is finitely generated, say with generators n_i, and $n_i = \sum f_{ij}(m_j)$ where $f_{ij} \in \mathrm{Hom}(M, N)$, then only images for those finitely many f_{ij} are needed to generate N. Hence there is a surjection $M^n \twoheadrightarrow N$. In particular, if we take $N = R$, R is projective, so the surjection splits and R is a summand of M^n.
In the other direction, if R is a summand of M^n, this implies M^n is a generator, and hence M is a generator also. □

Example 7.6: R is Morita equivalent to itself. In this case, take $P = R$ (the rank 1 free module), and $R = \mathrm{End}_R(R)^{\mathrm{op}}$. More generally, if we take $P = R^n$, then $S = \mathrm{End}_R(R^n)^{\mathrm{op}} = \mathrm{Mat}_n(R)$ is Morita equivalent to R also.
Using the lemma, we see that if R is Artinian with indecomposable projectives $P_1, \ldots, P_n$, $P = \bigoplus P_i^{m_i}$ is a projective generator iff $m_i \geqslant 1$ for all i. In particular, if we take $m_i = 1$ for all i, then $S = \mathrm{End}_R(P)^{\mathrm{op}}$ is what's known as a **based ring**, meaning that each irreducible L_i is a one-dimensional vector space over $D_i = \mathrm{End}_R(L_i)$.

Proposition 7.7: *Let $P = Re$ for an idempotent $e \in R$. Then P is a generator iff $R = ReR$.*

Proof. Suppose $R = ReR$. Then we can write $1 = \sum a_i e b_i$ for $a_i, b_i \in R$, so the map $P^n \to R$ given by $(x_1, \ldots, x_n) \mapsto \sum x_i b_i$ is onto. So by the lemma 7.5, P is a generator.
In the other direction, $M = R/ReR$ satisfies $\mathrm{Hom}(P, M) = eM = 0$, so if $M \neq 0$, P can't be a generator. □

7.2 Categories and the Yoneda Lemma

Quick review: a (small) category C consists of a set of objects $\mathrm{Ob}(C)$, a set of morphisms $\mathrm{Hom}_C(X, Y)$ for all $X, Y \in \mathrm{Ob}(C)$, an identity morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$, and an associative composition operation.

Remark 7.8: Small categories are those where $\mathrm{Ob}(C)$ is actually a set. Since there is no such thing as the "set of all sets", categories like Set or R-Mod are not small. We could get around this by fixing a universe and only considering sets from this universe. We could also consider "large" categories, whose objects form a collection more general than a set, called a class. We will ignore all these set-theoretic issues.

Given two categories C_1, C_2, we can talk about the category of functors $\mathrm{Fun}(C_1, C_2)$ whose objects are functors and whose morphisms are natural transformations.

Definition 7.9: *A functor F is **faithful** if the map* $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(F(X), F(Y))$ *is injective for all X, Y.*

Definition 7.10: *A functor F is **fully faithful** if the map* $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(F(X), F(Y))$ *is an isomorphism.*

Definition 7.11: *A functor F is **essentially surjective** if it is surjective on isomorphism classes of objects.*

Definition 7.12: *A functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ is an **equivalence of categories** if there exists $G: \mathcal{C}_2 \to \mathcal{C}_1$ such that $F \circ G, G \circ F$ are isomorphic to the respective identity functors (that is, they are naturally equivalent to the identity functors).*

Lemma 7.13: *A functor F is an **equivalence of categories** iff it is fully faithful and essentially surjective.*

Proof. Since we are ignoring set-theoretic considerations, we get to use the axiom of choice. It's clear that if F is an equivalence, then it's fully faithful and essentially surjective. In the other direction, if F is essentially surjective, the axiom of choice allows us to choose $X \in \operatorname{Ob}(\mathcal{C}_2)$ and $G(X) \in \operatorname{Ob}(\mathcal{C}_1)$ such that $i_X : X \cong F(G(X))$. Then we can define $G(f : X \to Y)$ as follows: first $i_Y^{-1} \circ f \circ i_X$ gives a map $F(G(X)) \to F(G(Y))$, and because F is fully faithful, this corresponds to a unique $G(f): G(X) \to G(Y)$. Then one can verify that G is indeed a functor and that $F \circ G$ and $G \circ F$ are equivalent to $\mathrm{id}_{\mathcal{C}_i}$. □

Lemma 7.14 (Yoneda Lemma): *For a category $\mathcal{C}$, consider the functors $R: \mathcal{C}^{\mathrm{op}} \to \operatorname{Fun}(\mathcal{C}, \mathrm{Set})$ and $C: \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ where $R(X): T \mapsto \operatorname{Hom}(X, T)$ and $C(X): T \to \operatorname{Hom}(T, X)$. Then R, C are fully faithful. Here R is for "represent" and C for "corepresent".*

Proof (Sketch). For $X, Y \in \operatorname{Ob}(\mathcal{C})$, there's a natural map $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(R(X), R(Y))$ given by composing with the map $X \to Y$. In the other direction, given $\varphi: R(X) \to R(Y)$, send it to the element $\varphi(\mathrm{id}_X) \in \operatorname{Hom}(X, Y)$. It's easy to see these are inverse bijections. The argument for C is similar. □

That is, an object in $\mathcal{C}$ is uniquely defined up to unique isomorphism up to the functor it (co)represents.

Example 7.15: The initial (resp. final) object of a category $\mathcal{C}$ is an object I (resp. F) such that $\operatorname{Hom}(I, X)$ (resp. $\operatorname{Hom}(X, F)$) is a singleton. By the Yoneda lemma, initial and final objects are unique up to unique isomorphism (if they exist). For example, in the category R-Mod, the zero module is both initial and final.

Definition 7.16: *The **coproduct** (resp. **product**) is the object representing (resp. corepresenting) the product of Hom sets:* $\operatorname{Hom}\left(\coprod X_i, T\right) = \prod \operatorname{Hom}(X_i, T)$ *and* $\operatorname{Hom}\left(T, \prod X_i\right) = \prod \operatorname{Hom}(T, X_i)$. *These are unique up to unique isomorphism if they exist.*

Example 7.17: In R-Mod, these both exist; coproduct is the direct sum and product is the usual product.

Remark 7.18: We can characterize the statement that a finite direct sum is the same as a finite product in categorical terms. Using the final object 0, there is a morphism $\coprod X_i \to X_i$. Hence, there is a map $\coprod X_i \to \prod X_i$, and this is an isomorphism when the X_i form a finite collection.

Remark 7.19: This can also be used to show that $\operatorname{Hom}(M, N)$ has an abelian group structure. You can define the sum of two maps $f, g: M \to N$ as the composition

$$M \xrightarrow{f \times g} N \times N \cong N \amalg N \xrightarrow{\mathrm{id}_N \amalg \mathrm{id}_N} N.$$

7.3 Proof of Morita equivalence theorem

Proof (of Theorem 7.3). Suppose $F\colon S\text{-Mod} \to R\text{-Mod}$ is an equivalence. We will show that $P := F(S)$ is a finitely generated projective generator in R-Mod and that $S = \mathrm{End}_S(S)^{\mathrm{op}} = \mathrm{End}_R(P)^{\mathrm{op}}$. This follows from the following observations:

- F sends projective S-modules to projective R-modules. M is projective iff $\mathrm{Hom}(M, -)$ is exact, i.e. sends a surjective map of modules to a surjective map of sets. A map of modules $T_1 \to T_2$ is surjective iff $\mathrm{Hom}(T_2, X) \hookrightarrow \mathrm{Hom}(T_1, X)$ is injective for all X. Using essential surjectivity of F, we find $N_1, N_2, Y \in S$-Mod such that $F(N_i) \cong T_i$ and $F(X) \cong Y$; then the full faithfulness of F implies that $N_1 \twoheadrightarrow N_2$. Then $\mathrm{Hom}(M, N_1) \twoheadrightarrow \mathrm{Hom}(M, N_2)$ combined with full faithfulness of F translates this into $\mathrm{Hom}(F(M), T_1) \twoheadrightarrow \mathrm{Hom}(F(M), T_2)$.
- F sends a projective generator to a projective generator, since $\mathrm{Hom}(M, N) = 0 \Leftrightarrow \mathrm{Hom}(F(M), F(N)) = 0$ by full faithfulness of F.
- F sends finitely generated projective S-modules to finitely generated projective R-modules. Use the following characterization of finitely generated projectives: a projective P is finitely generated iff $\mathrm{Hom}(P, -)$ commutes with arbitrary coproducts (i.e. $\coprod \mathrm{Hom}(P, X_i) = \mathrm{Hom}\left(P, \coprod X_i\right)$. If P is projective and finitely generated, it's a direct summand of S^n, which has this property, so P also has this property. In the other direction, suppose $\mathrm{Hom}(P, -)$ commutes with coproducts. We know P is the direct summand of some free module, say $\bigoplus_I S$, which then splits as $P \oplus Q$. Then $\mathrm{Hom}(P, \bigoplus_I S) = \bigoplus_I \mathrm{Hom}(P, S)$, so the image of $P \hookrightarrow \bigoplus_I S$ must land in a finite direct sum $S^n = \bigoplus_J S$, $|J| < \infty$. S^n will also split as $P \oplus (Q \cap S^n)$, so P is in fact finitely generated. Since F is an equivalence of categories, it preserves the property that $\mathrm{Hom}(F(P), -)$ commutes with arbitrary coproducts, so $F(P)$ is also finitely generated projective.

Combining these three, we get that $F(S)$ is a finitely generated projective generator. Because F is fully faithful, $\mathrm{Hom}_S(S, S) \cong \mathrm{Hom}_R(F(S), F(S)) = \mathrm{End}_R(P)$, so $S = \mathrm{End}_R(P)^{\mathrm{op}}$.

In the other direction, we want to show that if $S = \mathrm{End}_R(P)^{\mathrm{op}}$ for P a finitely generated projective generator P of R-Mod, the functor $F_P\colon M \mapsto \mathrm{Hom}_R(P, M)$ is the desired equivalence of categories. Here $M \in R$-Mod and $\mathrm{Hom}_R(P, M)$ has an S-action via composition.

F_P induces an isomorphism $\mathrm{Hom}_R(P, N) \cong \mathrm{Hom}_S(F_P(P), F_P(N))$ for all N: the RHS will be $\mathrm{Hom}_S(S, F_P(N)) \cong F_P(N) = \mathrm{Hom}(P, N)$. This isomorphism coincides with the F_P-action on morphisms.

Since P is finitely generated and projective, F_P commutes with coproducts. Moreover, P is a projective generator, we claim we can find an exact sequence $P^{\oplus J} \to P^{\oplus I} \to M \to 0$.

Lemma 7.20: *A projective module P is a generator iff the free module R is a direct summand in P^n for some n iff every module is a quotient of $P^{\oplus I}$.*

Now we want to show that $\mathrm{Hom}(M, N) \to \mathrm{Hom}(F_P(M), F_P(N))$ is an isomorphism. Notice that if this is true for M_1, M_2, it's also true for $\mathrm{coker}(f)$, $f\colon M_1 \to M_2$ because exactness of F_P implies that both Hom-spaces are the kernel of the map $\mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N)$. So by the above, it suffices to show that this is true for $M = P^{\oplus I}$, but that is what we proved above. So F_P is fully faithful.

To see that F_P is essentially surjective, take $N \in S$-Mod, which fits in an exact sequence $S^{\oplus J} \xrightarrow{f} S^{\oplus I} \to N \to 0$. Because F_P is fully faithful, $f = F_P(g)$ for $g\colon P^{\oplus J} \to P^{\oplus I}$. Hence $N \cong F_P(\mathrm{coker}(g))$. Thus, F_P is an equivalence of categories. □

Example 7.21: Now it's interesting to consider notions that are invariant under Morita equivalence. We will see that the center $Z(R)$ and cocenter $C(R)$ of a ring are such notions, i.e. if R, S are Morita equivalent, they have the same center and the same cocenter.