Lecture 8: Morita Theory Continued: (co)Centers, Functors and Bimodules

8.1 Center and cocenter

Last time, we claimed that the center and cocenter are Morita invariant notions. Recall that the center $Z(R)$ is defined as

$$Z(R) := \{z \in R \mid zr = rz \ \forall r \in R\}$$

and is a commutative subring in R. The cocenter $C(R)$ is

$$C(R) := R / \sum_i [y_i, x_i],$$

i.e. the quotient of R by combinations of the commutators of elements in R. $C(R)$ is an abelian group and an $Z(R)$-module, but generally does not have a ring structure.

Proposition 8.1: *If $R \sim_M S$, then $Z(R) \cong Z(S)$, $C(R) \cong C(S)$.*

Remark 8.2: We will see later that $Z(R) = \mathrm{HH}^0(R)$, the 0th Hochschild cohomology, and $C(R) = \mathrm{HH}_0(R)$, the 0th Hochschild homology, and that the ith Hochschild (co)homology is also Morita invariant.

Proof. We will need several intermediate lemmas that allow us to describe $Z(R)$ and $C(R)$ purely in terms of the category of modules.

Lemma 8.3: $Z(R) \cong \mathrm{End}(\mathrm{Id}_R)$, *i.e. endomorphisms of the identity functor in R-Mod, as commutative rings.*

Proof. An element in $\mathrm{End}(\mathrm{Id}_R)$ is a collection of maps $z_M \in \mathrm{End}(M)$ such that $z_N \circ f = f \circ z_M$ for all $f\colon M \to N$ maps of R-modules. If we take a central element $z \in R$, it corresponds to the functor where z_M is just the left action of z on M. If we are given a collection z_M, consider z_R; note that it must commute with left multiplication by r for all $r \in R$, so it must be a central element. Hence we get the desired isomorphism. □

Definition 8.4: *Let Proj_R be the category of finitely generated projective R-modules. A **trace map** for Proj_R with values in an abelian group A is an assignment of an element $\tau(P, \varphi) \in A$ for every $P \in \mathrm{Ob}(\mathrm{Proj}_R)$, $\varphi \in \mathrm{End}(P)$, such that*

$$\begin{aligned} \tau(P \oplus Q, \varphi \oplus \psi) &= \tau(P, \varphi) + \tau(Q, \psi) \\ \tau(P, a \circ b) &= \tau(Q, b \circ a), a\colon Q \to P, b\colon P \to Q. \end{aligned}$$

Lemma 8.5: *Let Proj_R be the category of finitely generated projective R-modules. Then $C(R)$ is the **universal abelian group receiving a trace map** for Proj_R. In other words, $C(R)$ is isomorphic (as abelian groups) to the quotient of the free abelian group generated by pairs (P, φ) by the relations $(P \oplus Q, \varphi \oplus \psi) - (P, \varphi) - (Q, \psi)$ and $(P, a \circ b) - (Q, b \circ a)$ (where $a\colon Q \to P$, $b\colon P \to Q$).*

Proof. Let us restate this in terms of matrices. Let $A = (a_{ij}) \in \mathrm{Mat}_n(R)$ and set $\overline{\mathrm{Tr}}(A) = \sum a_{ii} \pmod{[R, R]}$. Then

$$\overline{\mathrm{Tr}}(AB) = \overline{\mathrm{Tr}}(BA).$$

Call the abelian group in the statement $\tilde{C}(R)$. We will use $\overline{\mathrm{Tr}}$ to construct an isomorphism $\tau\colon \tilde{C}(R) \to C(R)$. Let P be a finitely generated projective. Then it's the summand of a free, so choose Q, n such that $P \oplus Q = R^n$. Then $(\varphi \oplus 0) \in \mathrm{End}(R^n)$ with matrix A_φ. Set

$$\tau(P, \varphi) := \overline{\mathrm{Tr}}(A_\varphi).$$

Then τ is independent of choices of Q, n and satisfies $\tau(P, ab) = \tau(Q, ba)$. Also, τ is clearly additive on direct sums. So τ is a homomorphism.

It is onto since we can choose $P = R$ and φ multiplication by any element in R. To see it's injective, it suffices to show that $(R^n, A) = (R, \sum a_{ii})$ in $\tilde{C}$. But this is true because a matrix with zero sum of diagonal elements will map to 0 in $C(\mathrm{Mat}_n(R))$. □

Therefore, center and cocenter depend only on the category R-Mod, which shows they are Morita invariant. □

Example 8.6: For $a \in R$, we can consider the operator $R \to R$ of right multiplication by a. The trace of this map is just $[a] \in C(R)$.

8.2 Morita equivalence via functors and bimodules

Definition 8.7: *Let R, S be rings. An R, S-bimodule M is an abelian group carrying a commuting left action of R and right action of S (i.e. a left S^{op} action). We denote such a module by ${}_RM_S$.*

Given a bimodule ${}_RP_S$, we get a functor $F_P\colon S\text{-Mod} \to R\text{-Mod}$ given by $M \mapsto P \otimes_S M$. It is easy to see that $F_Q \circ F_P = F_{Q \otimes_S P}$ for bimodules ${}_RQ_S$ and ${}_SP_T$. Thus, we have a functor from R, S-Bimod $\to$ Fun $(S\text{-Mod}, R\text{-Mod})$.

Lemma 8.8: *The functor $P \mapsto F_P$ is fully faithful.*

Proof. There is a natural map $\mathrm{Hom}(P, Q) \to \mathrm{Hom}(F_P, F_Q)$. To construct a map in the other direction, note that $P = F_P(S)$, and this is an isomorphism of R, S-bimodules because the right action of S on $F_P(S)$ is obtained by applying F_P to $\mathrm{End}(S)$. This defines a map $\mathrm{Hom}(F_P, F_Q) \to \mathrm{Hom}(P, Q)$, and you can check that it's the inverse bijection to the first map. □

Remark 8.9: In the proof of the Morita equivalence theorem last time, we used the functor $M \mapsto \mathrm{Hom}_R(P, M)$. This can be written as $M \mapsto \tilde{P} \otimes_R M$, where $\tilde{P} = \mathrm{Hom}_R(P, R)$ as a right R-module. We could rewrite $\mathrm{End}_R(P)^{\mathrm{op}} = \mathrm{End}_{R^{\mathrm{op}}}(\tilde{P})$. In fact, $P \mapsto \tilde{P}$ gives an equivalence of categories $\mathrm{Proj}_R^{\mathrm{op}} \to \mathrm{Proj}_{R^{\mathrm{op}}}$.

Remark 8.10: Recall that in an equivalence of categories, you have two functors F, G and $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$, $F \circ G \simeq \mathrm{Id}_{\mathcal{D}}$. It turns out that if you fix F, G, and the first isomorphism of functors, then the second isomorphism of functors is uniquely determined so that if the two isomorphisms $F \circ G \circ F \simeq F$ coincide (from either $F \circ \mathrm{Id}_{\mathcal{C}}$ or $\mathrm{Id}_{\mathcal{D}} \circ F$, the two isomorphisms $G \circ F \circ G \simeq G$ also coincide.

Therefore, if we want to define a Morita equivalence between A, B, we can rephrase this as finding ${}_AP_B, {}_BQ_A$, which will give us two functors A-Mod $\to$ B-Mod and B-Mod $\to$ A-Mod, such that $P \otimes Q \simeq A$ and $Q \otimes P \simeq B$, i.e. their compositions are isomorphic to the respective identity functors.

Definition 8.11: *A **Morita context** is the data of $A, B, {}_A P_B, {}_B Q_A$ with maps $\tau\colon P \otimes_B Q \to A$ and $\eta\colon Q \otimes_A P \to B$ such that the two arrows $P \otimes_B Q \otimes_A P \to P$ coincide and likewise for $Q \otimes_A P \otimes_B Q \to Q$.*
This can be rewritten in matrix form: τ, η, and the bimodule structures define multiplication on matrices of the form

$$\begin{pmatrix} a & p \\ q & b \end{pmatrix}, \; a \in A, p \in P, q \in Q, b \in B.$$

We can now talk about pq, as $\tau(p, q)$, etc., and our compatibility condition means this matrix multiplication is associative.

Example 8.12: Let B be a ring and $M \in B$-Mod. The **derived Morita context** is given by $A = \operatorname{End}_B(M)^{\mathrm{op}}$, $Q = M$, $P = \operatorname{Hom}_B(M, B)$, and $\tau(p \otimes q) = p(q)$, $\eta(q \otimes p)\colon m \mapsto p(m)q$.
We can verify that the arrows $P \otimes_B Q \otimes_A P \to P$ coincide: $p \otimes q \otimes p' \mapsto p(q) \otimes p'$, which sends $m \mapsto p'(m)p(q)$. The other map is $p \otimes q \otimes p' \mapsto p \otimes \eta(q \otimes p')$, which sends $m \mapsto p(p'(m)q) = p'(m)p(q)$. A similar argument holds for $Q \otimes_A P \otimes_B Q \to Q$.

Theorem 8.13: *For a derived Morita context, the functors given by P, Q are inverse equivalences iff M is a finitely generated projective generator.*

This is a reformulation of the theorem we proved last time. The proof is a consequence of the below lemmas.

Definition 8.14: *A **generator** $M \in R$-Mod is an object such that $\operatorname{Hom}_R(M, -)$ is faithful.*

Lemma 8.15: *M is a generator iff for all N, there exists a surjection $M^{\oplus I} \twoheadrightarrow N$, iff R is a direct summand of M^n.*

Lemma 8.16: *For a derived Morita context,*
a) *$\tau\colon P \otimes_B Q \to A$ is onto iff $Q = M$ is a generator over A.*
b) *$\eta\colon Q \otimes_A P \to B$ is onto iff $Q = M$ is a finitely generated projective over A.*

Proof. By definition $\operatorname{im}(\tau)$ is the sum of images of all homomorphisms $M \to A$. So τ is onto exactly when the sum of the images is A, which is when M is a generator. This proves a).
For b), first suppose η is onto. Then $1_B = \operatorname{Id}_M = \sum_{i=1}^n e_i f_i$ where $f_i\colon M \to A$ and $e_i\colon A \to M$. Then consider the maps $m \mapsto (f_1(m), \dots, f_n(m))$ and $(a_1, \dots, a_n) \mapsto \sum a_i e_i$. Their composition $M \to A^n \to M$ is the identity, so M is a direct summand of A^n, implying it's a finitely generated projective.
In the other direction, suppose that M is a finitely generated projective. Then write $M = A^n e$ for an idempotent e. Then $\operatorname{End}(M) = e \operatorname{Mat}_n(A) e$ and we have a surjection $A^n e \otimes eA^n \to \operatorname{End}(M)$. ◻

Lemma 8.17: *In a Morita context, τ (resp. η) is onto implies τ (resp. η) is an isomorphism.*

Proof. Suppose that $\tau\colon P \otimes_B Q \to A$ is onto. Then write $1 = \tau\left(\sum p_i \otimes q_i\right)$. Consider the map

$$Q \to Q \otimes_A P \otimes_B Q, \; q \mapsto q \otimes \left(\sum p_i \otimes q_i\right).$$

Then the composition

$$Q \to Q \otimes_A P \otimes_B Q \xrightarrow{\eta \otimes \mathrm{id}} B \otimes_B Q = Q$$

is the identity map. Tensoring with P on the left, we get the identity map $P \otimes_B Q \to P \otimes_B Q$. But the composition is also equal to

$$P \otimes_B Q \to (P \otimes_B Q) \otimes_A P \otimes_B Q \xrightarrow{\tau \otimes \mathrm{id}} A \otimes_A P \otimes_B Q = P \otimes_B Q$$

where the first arrow sends $p \otimes q \mapsto p \otimes q \otimes \left(\sum p_i \otimes q_i\right)$. Since an element in $\ker \tau$ would be killed by this composition, we must have $\ker \tau = 0$, so τ is an isomorphism. A similar argument works for η. ◻

8.3 Serre quotients

Motivating question: suppose that $P \in A$-Mod is a finitely generated projective but not a generator and $B = \mathrm{End}_A(P)^{\mathrm{op}}$. How are A-Mod and B-Mod related? It turns out that B-Mod is a Serre quotient of A-Mod by $\{M | \mathrm{Hom}(P, M) = 0\}$.

Definition 8.18: *A **Serre subcategory** of an abelian category (defined next time) is a full subcategory closed under subquotients and extensions. That is, for an SES $0 \to M_1 \to M \to M_2 \to 0$, M is n the subcategory iff M_1, M_2 are.*

Example 8.19: A Serre subcategory in the category of finite length modules is uniquely determined by the set of irreducible objects it contains. So such subcategories are in bijection with subsets of the set of isomorphism classes of irreducibles.

Let $\mathcal{A}$ be a Serre subcategory of an abelian category and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory.

Definition 8.20: *A homomorphism $f : M \to N$ is an **isomorphism modulo** $\mathcal{B}$ if $\ker(f), \mathrm{coker}(f) \in \mathcal{B}$.*

Definition 8.21: *The **Serre quotient** $\mathcal{A}/\mathcal{B}$ is the category with a universal functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ sending isomorphisms modulo $\mathcal{B}$ to isomorphisms. (That is, for any functor $\mathcal{A} \to C$ sending isos modulo $\mathcal{B}$ to isos, there's a unique functor $\mathcal{A}/\mathcal{B} \to C$ making the diagram commute.)*

The Serre quotient has the same objects as $\mathcal{A}$, but different Hom-sets.

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