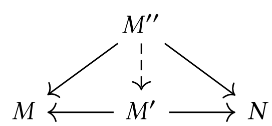
Lecture 9: More on Serre Quotients, Abelian Categories

9 March 9 - more on Serre quotients, abelian categories

9.1 More on Serre quotients

Let $\mathcal{A}$ be a Serre subcategory in R-Mod and $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. We defined the Serre quotient abstractly, but here is a more concrete description:

- Objects of $\mathcal{A}/\mathcal{B}$ are objects of $\mathcal{A}$.
- The morphisms $\operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(M, N)$ are equivalence classes of "roof diagrams" $M \leftarrow M' \rightarrow N$, where the left arrow $M \leftarrow M'$ is an isomorphism modulo $\mathcal{B}$ (i.e. its kernel and cokernel are both in $\mathcal{B}$). Two roof diagrams $M \leftarrow M' \rightarrow N$ and $M \leftarrow M'' \rightarrow N$ are equivalent if there exists a map $M'' \rightarrow M'$ commuting with the other arrows, i.e.



  Another way to phrase this:

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) = \operatorname*{colim}_{M' \to M} \operatorname{Hom}(M', N)$$

  where the colimit is taken over the category of objects $M' \in \mathcal{A}$ equipped with isomorphisms modulo $\mathcal{B}$ to M.

Remark 9.1: We could also phrase $\operatorname{Hom}_{\mathcal{A}/\mathcal{B}}$ in terms of "lower roof" diagrams, where the arrows are reversed, so $\operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(M, N)$ consists of diagrams $M \rightarrow N' \leftarrow N$ where $N' \leftarrow N$ is an isomorphism modulo $\mathcal{B}$. Why are these definitions equivalent? Given a lower roof diagram, you can construct the upper roof by setting $M' := M \times_{N'} N$ (the pullback), i.e. $\ker(M \oplus N \rightarrow N')$. Given an upper roof diagram, you can set N' to be the pushforward, namely $N' := \operatorname{coker}(M' \rightarrow M \oplus N)$.

Example 9.2: Let $\mathcal{A}$ be the category of finite length modules over an Artinian algebra R and $\mathcal{B}$ be the subcategory of modules that do not have some fixed irreducibles $L_1, \ldots, L_i$ in their Jordan-Holder series. Then $\mathcal{A}/\mathcal{B}$ will be the category of finite length modules over

$$S = \mathrm{End}\left(\bigoplus_{j=i+1}^{n} P_j\right)^{\mathrm{op}},$$

i.e. the sum of the projective covers of the remaining irreducibles.
If we remove the finite length assumption, then you get a special case of

$$R\text{-Mod}/(P^{\perp}) \cong \mathrm{End}_R(P)^{\mathrm{op}}\text{-Mod}.$$

Example 9.3: Let R be a commutative ring, $\mathcal{A} = R$-Mod, $I \subset R$, and $\mathcal{B}$ be the modules where every element of I acts locally nilpotently. Then $\mathcal{A}/\mathcal{B} = \mathrm{QCoh}(\mathrm{Spec}(R)\backslash Z_I)$ where Z_I is the zero set of I. This is a quasiaffine scheme (not necessarily affine).
You can also get (quasi)coherent sheaves on more general varieties using the Serre quotient. For example, a projective variety X over a field k can be obtained as $\mathrm{Proj}(A)$ for a positively graded commutative algebra A with $A_0 = k$. Then $\mathrm{Coh}(X)$ is the Serre quotient

$$A\text{-Mod}^{\mathrm{gr}}_{\mathrm{fg}}/A\text{-Mod}_0$$

where $A\text{-Mod}^{\mathrm{gr}}_{\mathrm{fg}}$ is the category of finitely generated graded modules and $A\text{-Mod}_0$ is the subcategory of finite-dimensional (equivalently, concentrated in finitely many degrees) modules. Geometrically, this corresponds to starting with dilation equivariant sheaves on the cone $\mathrm{Spec}(A)$ and throwing away the origin.

9.2 Adjoint functors and (co)limits

Definition 9.4: *An **adjunction** for a pair of functors $L\colon C_1 \to C_2$, $R\colon C_2 \to C_1$ is an isomorphism*

$$\mathrm{Hom}_{C_2}(L(X), Y) \cong \mathrm{Hom}_{C_1}(X, R(Y))$$

*that is functorial in X, Y. Then we say that L, R are **adjoint functors**, that L is the **left adjoint** of R, and R is the **right adjoint** of L.*
The Yoneda Lemma indicates that L determines R up to unique isomorphism and vice versa (if it exists).

Example 9.5: In general, free and forgetful functors are adjoint; for example, the functor sending a set S to the corresponding free structure (group, abelian group, module, algebra, etc.) on S is left adjoint to the forgetful functor to Set. Likewise, the functor sending a Lie algebra to its universal enveloping algebra $\mathfrak{g} \to U(\mathfrak{g})$ is left adjoint to the functor sending an associative algebra to itself but as a Lie algebra (with the Lie bracket $[x, y] = xy - yx$).

Example 9.6: It is possible for a functor to have a left adjoint but no right adjoint: for example, the full embedding of commutative rings into associative rings has a left adjoint sending R to the quotient by the 2-sided ideal generated by the commutators. But it has no right adjoint.

Example 9.7 (Tensor-Hom adjunction): Let ${}_AP_B$ be a bimodule. Then $L = P \otimes_B -\colon B\text{-Mod} \to A\text{-Mod}$ is left adjoint to $R = \mathrm{Hom}_A(P, -)\colon A\text{-Mod} \to B\text{-Mod}$.

Example 9.8: Consider the category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ of functors from $\mathcal{D} \to \mathcal{C}$. The functor

$$\mathrm{Cons}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}),\ \mathrm{Cons}(X)(Y) = X, \mathrm{Cons}(f) = \mathrm{Id}_X$$

has right adjoint Cons^*; this may or may not exist, but if it does, $\mathrm{Cons}^*(F)$ is the **limit** or **inverse limit** of F. Likewise, the left adjoint $^*\mathrm{Cons}$, if it exists, sends F to the **colimit** or **direct limit** of F.
We can describe the limit more concretely. $\mathrm{Cons}^*(F)$ has the following property: it is the universal object equipped with compatible maps into $F(i), i \in \mathcal{D}$. That is, given an object X with compatible maps into $F(i)$ for $i \in \mathcal{D}$, there is a unique map $X \to \mathrm{Cons}^*(F)$ that makes the diagram commute.
Note that not all limits and colimits may exist in a category. Some examples of limits and colimits: the colimit of $\bullet \quad \bullet$ is the coproduct, while its limit is the product. The limit of $\bullet \to \bullet \leftarrow \bullet$ is the pullback, the colimit of $\bullet \leftarrow \bullet \to \bullet$ is the pushout.

9.3 Additive categories

We are interested in categories like R-Mod that have additional structure.

Definition 9.9: *An* ***additive category*** $\mathcal{A}$ *is a category where each Hom set has the structure of an abelian group such that the composition is bilinear and the following properties hold:*

a) *There exists an object* $0_{\mathcal{A}}$ *such that* $\mathrm{Hom}(0_{\mathcal{A}}, 0_{\mathcal{A}}) = 0$ *(the zero group),*
b) *For every* $M_1, M_2 \in \mathcal{A}$, *there exists an object* $S = M_1 \oplus M_2$ *with morphisms* $p_i\colon S \to M_i$ *and* $\iota_i\colon M_i \to S$ *such that* $p_1\iota_2 = p_2\iota_1 = 0$, $p_1\iota_1 = \mathrm{id}_{M_1}, p_2\iota_2 = \mathrm{id}_{M_2}$, *and* $\iota_1 p_1 + \iota_2 p_2 = \mathrm{id}_S$.

This implies that $\mathrm{Hom}(0, M) = \mathrm{Hom}(M, 0) = 0$, so 0 is both the initial and final object. Also, S is both the coproduct and product of M_1, M_2: you can see this by noting that the corresponding fact is true for abelian groups, then apply this to $\mathrm{Hom}(S, X)$ and $\mathrm{Hom}(X, S)$.

Notice that we were able to deduce a global property (about Hom in every object) from a local property (only looking at $M, N, S, 0$).

Note : We don't need to include an addition on Hom sets in the definition. If we know that there is an initial and final object and that therefore, the resulting map from coproducts to products is an isomorphism, you can recover addition on Hom sets, as discussed in the category of modules. But it's more convenient to list it in the definition.

9.4 Abelian categories

An abelian category is essentially a "category where you can do homological algebra" and was introduced by Grothendieck.

Definition 9.10: *An **abelian category** is an additive category satisfying*

- *AB1: existence of kernel and cokernels: that is, objects representing the functor* $X \to \ker(\mathrm{Hom}(X, M) \to \mathrm{Hom}(X, N))$ *and corepresenting the functor* $X \to \ker(\mathrm{Hom}(N, X) \to \mathrm{Hom}(M, X))$ *for a morphism* $f: M \to N$*. Morphisms with zero kernel are **monic** and morphisms with zero cokernel are **epic**.*
- *AB2: A monic morphism is a kernel; that is, for* $f: M \to N$*, let* $K = \ker f$ *and* $C = \operatorname{coker} f$*, then* $\operatorname{coker}(K \to M) \to \ker(N \to C)$ *is an isomorphism.*

One can also add the additional axioms

- *AB3: the existence of arbitrary coproducts*
- *AB4: the coproduct of any family of monic morphisms is monic*

A subobject of A *is an object* A_i *with a monic morphism* $A_i \hookrightarrow A$*. The sum of some subobjects* A_i *is* $\operatorname{im}\left(\coprod A_i \to A\right)$*. The intersection of two subobjects* A, B *of* C *is* $\ker(C \to C/B \oplus C/A)$*. We can add one last axiom*

- *AB5:* $(\sum A_i) \cap B = \sum (A_i \cap B)$ *for a collection of increasing subobjects* A_i *in* A*.*

We can also define AB3,4,5: a category satisfies ABn* if* $\mathcal{A}^{\mathrm{op}}$ *satisfies ABn.*
*If a category satisfies AB1-5, it's called a **Grothendieck category**.*

Definition 9.11: *A category* $\mathcal{D}$ *is **filtered** if* $\mathrm{Ob}(\mathcal{D}) \neq \varnothing$ *and for all* $a, b \in \mathcal{D}$*, there exists* $c \in \mathcal{D}$ *such that* $\mathrm{Hom}(a, c), \mathrm{Hom}(b, c)$ *are nonempty and such that for every pair of parallel morphisms* $e, f: a \to b$*, there exists* $g: b \to c$ *such that* $ge = gf$*.*

The key feature of Grothendieck categories is that filtered colimits exist and are exact.

Remark 9.12: The category of R-modules satisfies AB5, AB3*, and AB4*.

Remark 9.13: The only abelian category satisfying AB3-5 and AB3*-5* is the zero category. Sketch of proof: consider an object X in such a category and let Σ, Π be the coproduct and product of countably many copies of X. There is a canonical map $c: \Sigma \to \Pi$; it is monic because it's the colimit of embeddings of a direct summand and epic since it is the inverse limit of surjections to a direct summand. Hence c is an isomorphism. Now consider the composition φ of the arrows $X \to \Pi \xrightarrow{c^{-1}} \Sigma \to X$ where the first arrow is the diagonal and the second arrow is the codiagonal. Then one can check that $\varphi + \mathrm{id}_X = \mathrm{id}_X$ because "$\infty + 1 = \infty$". Hence $\mathrm{id}_X = 0$ and $X \cong 0$.

9.5 Compact projective generators and Serre quotients revisited

Definition 9.14: *An object* M *is **compact** if* $\mathrm{Hom}(M, -)$ *commutes with filtered colimits.*

If M is projective, this follows from commuting with arbitrary direct sums, since

$$\operatorname{colim}(F) = \operatorname{coker}\left(\bigoplus_{e:\, a \to b} F(a) \to \bigoplus_{a} F(a)\right)$$

where the morphism takes $x \mapsto x - F(e)(x)$. In general, this is not true, though it is true that every compact module is finitely generated.

Definition 9.15: *An object* P *is a **generator** if* $T \mapsto \mathrm{Hom}(P, T)$ *is a faithful functor. For a projective object, this is equivalent to the Definition 7.4. Alternatively, we could say that if* $P^{\perp}$ *is the full subcategory whose objects are* M *such that* $\mathrm{Hom}(P, M) = 0$*, then a projective object* P *is a generator iff* $P^{\perp} \cong \{0\}$*.*

Theorem 9.16: *An abelian category with coproducts (satisfying AB3) and a projective compact generator is* $\mathrm{End}(P)^{\mathrm{op}}$*-Mod where* P *is a projective compact generator.*

The proof is the same as the proof in the Morita theory case.

Corollary 9.17: *Let P be a compact projective object in an AB3 abelian category $\mathcal{A}$. Let $\mathcal{B} = P^{\perp}$. Then $\mathcal{A}/\mathcal{B} \cong \mathrm{End}(P)^{\mathrm{op}}$-Mod.*

Proof (Sketch). It's clear that 1) P is projective in $\mathcal{A}/\mathcal{B}$ (use the lower roof diagram Homs) and 2) P is a generator (in $\mathcal{A}/\mathcal{B}$). $\mathcal{B}$ is closed under coproducts, so the projection functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ commutes with coproducts. Hence P is compact in $\mathcal{A}/\mathcal{B}$. □

This proves the claim at the beginning of Section 8.3.

References for this lecture include the original article [7], which still makes for excellent reading. Textbook expositions can be found in [9] and in the appendix to [11].

18.706 Noncommutative Algebra
Spring 2023