Lecture 10: Exts and Tors, Resolutions

10 March 14 - Exts and Tors, Resolutions

10.1 Ext and Tor

Definition 10.1: *Let M, N be objects in an abelian category. $\mathrm{Ext}^i(M, N)$ is the derived functor of Hom. Recall that Hom is left exact in the second argument and right exact in the first argument, so you can take either the right derived functor of $\mathrm{Hom}(M, -)$ or the left derived functor of $\mathrm{Hom}(-, N)$, and these are the same. Although the most useful formalism for this is the derived category, we can also work in a typical category.*

The key property of Ext is that it is the **universal delta functor**. Delta functors were introduced by Grothendieck in his Tohoku paper; essentially, they turn short exact sequences to long exact sequences. Given a short exact sequence $0 \to L \to M \to N \to 0$, a delta functor is a functorial family of (additive) functors T^i with boundary morphisms $\delta_i \colon T^i(N) \to T^{i+1}(L)$ such that $0 \to T^0(L) \to T^0(M) \to T^0(N) \xrightarrow{\delta_0} T^1(L) \to \cdots \to$ is exact. One can define morphisms of delta functors as families of natural transformations that commute with the boundary morphisms, and a delta functor is universal when giving a morphism to any other delta functor is equivalent to only giving the natural transformation in degree zero.

To show that something is a universal functor, it's enough to show that it's "effaceable" (in the language of Grothendieck), meaning that every element $\varphi \in \mathrm{Ext}^i(M, N), i > 0$ is killed by some injection $N \hookrightarrow N'$.

To actually compute Ext, we use projective and injective resolutions. A projective resolution of M is an exact sequence $\cdots \to P_2 \to P_1 \to M \to 0$ where the P_i are projective; these always exist for R-modules. $\mathrm{Ext}(M, N)$ is computed by applying $\mathrm{Hom}(-, N)$ to the resolution, removing M, and computing the cohomology of the resulting complex. You can also compute Ext using left injective resolutions of N, i.e. $0 \to N \to I_1 \to \cdots \to$ where I_i are injective.

Then, in this case, it's easy to see that Ext is effaceable – every N has an injection into an injective I, and every M receives a surjection from a projective P, so these maps efface all elements in $\mathrm{Ext}^i, i > 0$ because $\mathrm{Ext}^i(P, N) = \mathrm{Ext}^i(M, I) = 0$ for $i > 0$.

A better formal setting for this is the homotopy category of complexes $\mathcal{H}\wr(R)$. The morphisms in this category are defined as follows: for C_1, C_2 complexes in R-Mod, let $\mathrm{Hom}^\bullet(C_1, C_2)$ be the complex where $\mathrm{Hom}^i(C_1, C_2) = \prod_j \mathrm{Hom}(C_1^i, C_2^{i+j})$ and define $\mathrm{Hom}_{\mathcal{H}\wr(R)}(C_1, C_2) := H^0(\mathrm{Hom}^\bullet(C_1, C_2))$. $\mathrm{Hom}^\bullet$ has a differential, which is to take the supercommutator with d. That is, it consists of maps $f \colon C_1 \to C_2$ that commute with d modulo the equivalence that $f \sim g$ if $f - g = d_{C_2}h + hd_{C_1}$ where $h \colon C_1^i \to C_2^{i+1}$ is any collection of maps.

Exercise : There is a full embedding R-Mod $\to \mathcal{H}\wr(R)$ taking $M \mapsto P_M$, a projective resolution of M, which is unique up to unique isomorphism in $\mathcal{H}\wr(R)$. (That is, projective resolutions are "unique up to homotopy").

Let $\mathcal{H}\wr^0(R)$ be category of complexes of projectives in nonpositive degree with $H^i = 0, i < 0$ (so they are exact outside of degree 0). Then there is an equivalence $\mathcal{H}\wr^0(R) \to R$-Mod taking $C \mapsto H^0(C)$.

Remark 10.2: M is projective iff $\mathrm{Ext}^1(M,N) = 0$ for all N. If M is projective, it has projective resolution $0 \to M \to M \to 0$. If $\mathrm{Ext}^1(M,N) = 0$, then $0 \to N \to P \twoheadrightarrow M \to 0$ has a splitting for all N (either use the definition that Ext^1 is extensions or the long exact sequence), so M is projective.

10.2 Projective, injective, and homological dimension

Definition 10.3: *The **projective dimension** of a module M is*

$$\max\left\{i \mid \exists N \text{ s.t. } \mathrm{Ext}^i(M,N) \neq 0\right\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}.$$

If $0 \to M' \to P \twoheadrightarrow M \to 0$, $\mathrm{pdim}(M) = \mathrm{pdim}(M') + 1$ unless M is projective. This is because for $i \geqslant 1$, the LES says $0 \to \mathrm{Ext}^i(M',N) \to \mathrm{Ext}^{i+1}(M,N) \to 0$.
Alternately, we can define projective dimension as the length of the minimal projective resolution. For example, if $\mathrm{pdim}(M) = 1$, that means $0 \to Q \to P \to M \to 0$ is a resolution of M.

Definition 10.4: *The **injective dimension** of a module N is*

$$\max\left\{d \mid \exists M \text{ s.t. } \mathrm{Ext}^d(M,N) \neq 0\right\}$$

or the length of the minimal injective resolution.

Definition 10.5: *The **homological dimension** of a ring R is the maximal projective dimension of an R-module, which is the same as the maximal injective dimension of an R-module. It is also*

$$\max\left\{d \mid \exists M,N \text{ s.t. } \mathrm{Ext}^d(M,N) \neq 0\right\}.$$

Remark 10.6: N is injective iff $\mathrm{Ext}^1(M,N) = 0$ for all cyclic M.

Proof. We'll show that if $0 \to M' \hookrightarrow M, M' \to N$, we can extend this to a map $M \to N$. By Zorn's Lemma, it suffices to show that it can be extended to some $M'' \subset M$ with $M' \subsetneq M''$. Pick $m \in M \setminus M'$ and let M'' be the submodule generated by M', m. We have an exact sequence $0 \to M \to M'' \to M''/M' \to 0$ and by construction M''/M' is cyclic.
Hence, if $\mathrm{Ext}^1(M''/M',N) = 0$, there exists an extension of $M' \to N$ to $M'' \to N$. □

This remark implies that

$$\mathrm{hdim}(R) = \max\left\{d \mid \mathrm{Ext}^d(M,N) \neq 0 \text{ for some } M,N \text{ s.t. } M \text{ is f.g.}\right\}.$$

Example 10.7: If R is left Noetherian, a finitely generated M has a resolution of finitely generated projectives. Then $\mathrm{Ext}^i(M,-)$ commutes with filtered colimits. Hence, we can assume that N is also finitely generated in the above definition of homological dimension. If R is Artinian, we can say more: it suffices to consider only irreducible M, N.

10.3 Cartan matrices

In this section suppose that R is Artinian and R-Mod refers only to finitely generated modules. If R has finite homological dimension, then K^0 (R-Mod) (Definition 3.17) is generated by classes of projective modules: for every simple, write a projective resolution $0 \to P_L^i \to \cdots \to P_L^0 \to L \to 0$, then $[L] = \sum(-1)^i P_L^i$.

Definition 10.8: *Let $L_1, \ldots, L_n$ be the irreducibles for a ring R, and $P_1, \ldots, P_n$ be their projective covers. The **Cartan matrix** of R is the $n \times n$ matrix with $C_{ij} = [P_j : L_i]$, the multiplicity of L_i in P_j.*
If R is finite-dimensional over an algebraically closed field, we can also say that $C_{ij} = \dim_k \mathrm{Hom}(P_i, P_j)$.

We then get an identification $K^0\,(R\text{-Mod}) \cong \mathbb{Z}^n$ via $[M] \mapsto ([M : L_1], \ldots, [M : L_n])$. Hence, if R has finite homological dimension, $C \in \mathrm{GL}_n(\mathbb{Z})$; the ijth entry of C^{-1} is

$$\sum_d (-1)^d \# \left\{ \text{summands of } P^d_{L_i} \text{ isomorphic to } P_j \right\}.$$

Corollary 10.9: *If $n = 1$, R has finite homological dimension iff $R = \mathrm{Mat}_n(D)$ for a skew field D.*

Now let R be Artinian and M a finitely generated module.

Lemma 10.10: *Let $\cdots \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} P^0 \to M \to 0$ be a projective resolution and set $C^i = \ker(d_i) = \mathrm{im}(d_{i-1})$. Then TFAE:*

a) *$P^{-i-1} \xrightarrow{d_{-i-1}} C^{-i}$ is a projective cover for every i.*
b) *$L \otimes_R P^\bullet$ has 0 differential for all irreducible right R-modules L.*
c) *$\mathrm{Hom}_R(P^\bullet, L)$ has 0 differential for all irreducible L.*

Resolutions satisfying these properties are minimal. From a), if it exists, it is unique up to non-unique isomorphism because projective covers are unique. From c), we see that in the minimal resolution,

$$P^{-d} = \bigoplus_i P_i^{m_i^d}, \ \mathrm{Ext}^d(M, L_i) = D_i^{m_i^d}$$

where $m_i^d = \dim_{D_i}(\mathrm{Ext}^d(M, L_i))$.

Proof. $P \twoheadrightarrow M$ is a projective cover iff it induces an isomorphism $\mathrm{Hom}(M, L) \to \mathrm{Hom}(P, L)$ for all irreducibles L. First, $P \twoheadrightarrow M$ iff $\mathrm{Hom}(M, L) \hookrightarrow \mathrm{Hom}(P, L)$ for all irreducibles L. If $P \twoheadrightarrow M$, then by applying $\mathrm{Hom}(-, L)$, which is left exact, we see that $\mathrm{Hom}(M, L) \hookrightarrow \mathrm{Hom}(P, L)$. If the map $P \to M$ is not onto, then $\mathrm{coker}(P \to M)$ is nonzero finitely generated, so it has irreducible quotient L. Then $M \twoheadrightarrow L$ is in the kernel of $\mathrm{Hom}(M, L) \to \mathrm{Hom}(P, L)$, so this map is not injective.
If $P \twoheadrightarrow M$ is a projective cover, and there exists $P \to L$ that doesn't come from some $M \to L$, then $\ker(P \to L) \twoheadrightarrow M$, so the surjection is not essential, a contradiction. If $P \twoheadrightarrow M$ is not a projective cover, then there exists $Q \hookrightarrow P$ with $Q \twoheadrightarrow M$. Then P/Q has a simple quotient L, and the map $P \twoheadrightarrow P/Q \twoheadrightarrow L$ cannot come from a map $M \to L$: if it did, then $P \twoheadrightarrow M \to L$ should pull back to $Q \twoheadrightarrow M \to L$, but this is the zero map because it's also the composition $Q \hookrightarrow P \twoheadrightarrow P/Q \twoheadrightarrow L$, which is zero. Hence $\mathrm{Hom}(M, L) \to \mathrm{Hom}(P, L)$ is not surjective.
By definition $0 \to P^{-i-1} \to P^{-i} \to C^{-i+1} \to 0$. If $P^{-i} \to C^{-i+1}$ is a projective cover, then $\mathrm{Hom}(C^{-i+1}, L) \cong \mathrm{Hom}(P^{-i}, L)$, iff $\mathrm{Hom}(P^{-i}, L) \cong \mathrm{Hom}(P^{-i+1}, L)$. □

Remark 10.11: This generalizes to $\mathbb{Z}_{\geqslant 0}$-graded rings where A_0 is Artinian and A_d is finitely generated over A_0. A common setting where this appears is an algebra A over an algebraically closed field k where A_0 is semisimple and A_d is finite-dimensional over k. In this setting, there are still indecomposable projectives. In minimal resolutions, each term has finitely many generators in each degree. The graded irreducibles are concentrated in one degree (use that if M is a graded A-module, then $M_{\geqslant k} := \bigoplus_{i \geqslant k} M_i \subset M$ is a A-submodule of M for any $k \in \mathbb{Z}$). It follows that graded irreducible A-modules are annihilated by $A_{\geqslant 1}$ so they are just irreducible A_0-modules (up to a shift of grading).
If $A_0 = k$ is just a field, and for finitely generated (graded) M, we can consider its Poincare series $\sum_i \dim(M_i) t^i \in \mathbb{Z}((t))$. More generally, if A_0 is semisimple then one can consider series $P_M := \sum_i [M_i] t^i \in \mathbb{Z}((t))^n$ where n is the number of irreducibles for A_0 and $[M_i] \in K^0\,(A_0\text{-Mod}) \cong \mathbb{Z}^n$. The Cartan matrix C now lies in $\mathrm{GL}_n(\mathbb{Z}[[t]])$ instead of $\mathrm{Mat}_n(\mathbb{Z})$ (it is clear that $C \in \mathrm{Mat}_n(\mathbb{Z}[[t]])$ and $C(0) = \mathrm{Id}$ since A_0 is semisimple, it then follows that $C \in \mathrm{GL}_n(\mathbb{Z}[[t]])$). If L_i has finite homological dimension, and A is Noetherian then $C^{-1} \in \mathrm{Mat}_n(\mathbb{Z}[t])$.
For example, if $A = k[x]$, considered as a graded algebra with $\deg x = 1$, then $n = 1$, $L_1 = k$, $P_1 = k[x]$, so $C = \sum_{i=0}^{\infty} t^i = \frac{1}{1-t}$, and $C^{-1} = 1 - t$.