10 March 14 - Exts and Tors, Resolutions

10.1 Ext and Tor

**Definition 10.1:** Let $M, N$ be objects in an abelian category. $\text{Ext}^i(M, N)$ is the derived functor of $\text{Hom}$. Recall that $\text{Hom}$ is left exact in the second argument and right exact in the first argument, so you can take either the right derived functor of $\text{Hom}(M, -)$ or the left derived functor of $\text{Hom}(-, N)$, and these are the same. Although the most useful formalism for this is the derived category, we can also work in a typical category.

The key property of Ext is that it is the **universal delta functor.** Delta functors were introduced by Grothendieck in his Tohoku paper; essentially, they turn short exact sequences to long exact sequences. Given a short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$, a delta functor is a functorial family of (additive) functors $T_i$ with boundary morphisms $\delta_i: T^i(N) \rightarrow T^{i+1}(L)$ such that $0 \rightarrow T^0(L) \rightarrow T^0(M) \rightarrow T^0(N) \xrightarrow{\delta_0} T^1(L) \rightarrow \cdots$ is exact. One can define morphisms of delta functors as families of natural transformations that commute with the boundary morphisms, and a delta functor is universal when giving a morphism to any other delta functor is equivalent to giving the natural transformation in degree zero.

To show that something is a universal functor, it’s enough to show that it’s “effaceable” (in the language of Grothendieck), meaning that every element $\varphi \in \text{Ext}^i(M, N), i > 0$ is killed by some injection $N \rightarrowtail N'$.

To actually compute Ext, we use projective and injective resolutions. A projective resolution of $M$ is an exact sequence $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow M \rightarrow 0$ where the $P_i$ are projective; these always exist for $R$-modules. $\text{Ext}(M, N)$ is computed by applying $\text{Hom}(-, N)$ to the resolution, removing $M$, and computing the cohomology of the resulting complex. You can also compute Ext using left injective resolutions of $N$, i.e. $0 \rightarrowtail N \rightarrowtail I_1 \rightarrowtail \cdots$ where $I_i$ are injective.

Then, in this case, it’s easy to see that Ext is effaceable – every $N$ has an injection into an injective $I$, and every $M$ receives a surjection from a projective $P$, so these maps efface all elements in $\text{Ext}^i, i > 0$ because $\text{Ext}^i(P, N) = \text{Ext}^i(M, I) = 0$ for $i > 0$.

A better formal setting for this is the homotopy category of complexes $\mathcal{H}^i(R)$. The morphisms in this category are defined as follows: for $C_1, C_2$ complexes in $R$-Mod, let $\text{Hom}^*(C_1, C_2)$ be the complex where $\text{Hom}^i(C_1, C_2) = \prod_j \text{Hom}(C_1^j, C_2^{i+j})$ and define $\text{Hom}_{\mathcal{H}^i(R)}(C_1, C_2) = H^0(\text{Hom}^*(C_1, C_2))$. $\text{Hom}^*$ has a differential, which is to take the supercommutator with $d$. That is, it consists of maps $f: C_1 \rightarrow C_2$ that commute with $d$ modulo the equivalence that $f \sim g$ if $f - g = dc_2 + hdc_1$ where $h: C_1 \rightarrow C_1^{i+1}$ is any collection of maps.

**Exercise:** There is a full embedding $R$-Mod $\rightarrow \mathcal{H}^i(R)$ taking $M \mapsto P_M$, a projective resolution of $M$, which is unique up to unique isomorphism in $\mathcal{H}^i(R)$. (That is, projective resolutions are “unique up to homotopy”)

Let $\mathcal{H}^0(R)$ be the category of complexes of projectives in nonpositive degree with $H^i = 0, i < 0$ (so they are exact outside of degree 0). Then there is an equivalence $\mathcal{H}^0(R) \rightarrow R$-Mod taking $C \mapsto H^0(C)$. 

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Remark 10.2: $M$ is projective iff $\text{Ext}^1(M, N) = 0$ for all $N$. If $M$ is projective, it has projective resolution $0 \rightarrow M \rightarrow M \rightarrow 0$. If $\text{Ext}^1(M, N) = 0$, then $0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0$ has a splitting for all $N$ (either use the definition that $\text{Ext}^1$ is extensions or the long exact sequence), so $M$ is projective.

10.2 Projective, injective, and homological dimension

Definition 10.3: The **projective dimension** of a module $M$ is

$$\max \left\{ i \mid \exists N \text{ s.t. } \text{Ext}^i(M, N) \neq 0 \right\} \in \mathbb{Z}_{\geq 0} \cup \{ \infty \}.$$ 

If $0 \rightarrow M' \rightarrow P \rightarrow M \rightarrow 0$, $\text{pd}(M) = \text{pd}(M') + 1$ unless $M$ is projective. This is because for $i \geq 1$, the LES says $0 \rightarrow \text{Ext}^i(M', N) \rightarrow \text{Ext}^{i+1}(M, N) \rightarrow 0$.

Alternately, we can define projective dimension as the length of the minimal projective resolution. For example, if $\text{pd}(M) = 1$, that means $0 \rightarrow Q \rightarrow P \rightarrow M \rightarrow 0$ is a resolution of $M$.

Definition 10.4: The **injective dimension** of a module $N$ is

$$\max \left\{ d \mid \exists M \text{ s.t. } \text{Ext}^d(M, N) \neq 0 \right\}$$

or the length of the minimal injective resolution.

Definition 10.5: The **homological dimension** of a ring $R$ is the maximal projective dimension of an $R$-module, which is the same as the maximal injective dimension of an $R$-module. It is also

$$\max \left\{ d \mid \exists M, N \text{ s.t. } \text{Ext}^d(M, N) \neq 0 \right\}.$$

Remark 10.6: $N$ is injective iff $\text{Ext}^1(M, N) = 0$ for all cyclic $M$.

Proof. We’ll show that if $0 \rightarrow M' \hookrightarrow M, M' \rightarrow N$, we can extend this to a map $M \rightarrow N$. By Zorn’s Lemma, it suffices to show that it can be extended to some $M'' \subseteq M$ with $M' \subseteq M''$. Pick $m \in M \setminus M'$ and let $M''$ be the submodule generated by $M', m$. We have an exact sequence $0 \rightarrow M \rightarrow M'' \rightarrow M''/M' \rightarrow 0$ and by construction $M''/M'$ is cyclic.

Hence, if $\text{Ext}^1(M''/M', N) = 0$, there exists an extension of $M' \rightarrow N$ to $M'' \rightarrow N$.

This remark implies that

$$\text{hdim}(R) = \max \left\{ d \mid \exists M, N \text{ s.t. } \text{Ext}^d(M, N) \neq 0 \text{ for some } M, N \text{ s.t. } M \text{ is f.g.} \right\}.$$ 

Example 10.7: If $R$ is left Noetherian, a finitely generated $M$ has a resolution of finitely generated projectives. Then $\text{Ext}^i(M, -)$ commutes with filtered colimits. Hence, we can assume that $N$ is also finitely generated in the above definition of homological dimension. If $R$ is Artinian, we can say more: it suffices to consider only irreducible $M, N$.

10.3 Cartan matrices

In this section suppose that $R$ is Artinian and $R$-Mod refers only to finitely generated modules. If $R$ has finite homological dimension, then $K^0 (R$-Mod) (Definition 3.17) is generated by classes of projective modules: for every simple, write a projective resolution $0 \rightarrow P^i_L \rightarrow \cdots \rightarrow P^0_L \rightarrow L \rightarrow 0$, then $[L] = \sum (-1)^i P^i_L$.

Definition 10.8: Let $L_1, \ldots, L_n$ be the irreducibles for a ring $R$, and $P_1, \ldots, P_n$ be their projective covers. The **Cartan matrix** of $R$ is the $n \times n$ matrix with $C_{ij} = [P_j : L_i]$, the multiplicity of $L_i$ in $P_j$.

If $R$ is finite-dimensional over an algebraically closed field, we can also say that $C_{ij} = \dim_k \text{Hom}(P_i, P_j)$.
We then get an identification $K^0(R\text{-Mod}) \cong \mathbb{Z}^n$ via $[M] \mapsto ([M : L_1], \ldots, [M : L_n])$. Hence, if $R$ has finite homological dimension, $C \in \text{GL}_n(\mathbb{Z})$; the $i$th entry of $C^{-1}$ is

$$\sum_d (-1)^d \# \{\text{summands of } P^d_L \text{ isomorphic to } P_j \}.$$ 

**Corollary 10.9:** If $n = 1$, $R$ has finite homological dimension iff $R = \text{Mat}_n(D)$ for a skew field $D$.

Now let $R$ be Artinian and $M$ a finitely generated module.

**Lemma 10.10:** Let $\cdots \xrightarrow{d_{i+1}} P^{-1} \xrightarrow{d_i} P^0 \to M \to 0$ be a projective resolution and set $C^i = \ker(d_i) = \text{im}(d_{i-1})$. Then TFAE:

a) $P^{-i-1} \xrightarrow{d_{i+1}} C^{-i}$ is a projective cover for every $i$.

b) $L \otimes R P^i$ has 0 differential for all irreducible right $R$-modules $L$.

c) $\text{Hom}_R(P^i, L)$ has 0 differential for all irreducible $L$.

Resolutions satisfying these properties are minimal. From a), if it exists, it is unique up to non-unique isomorphism because projective covers are unique. From c), we see that in the minimal resolution,

$$P^{-d} = \bigoplus_i P^d_i, \text{ Ext}^d(M, L_i) = D_i^{m_i}$$

where $m^d_i = \text{dim}_{D_i}(\text{Ext}^d(M, L_i))$.

**Proof.** $P \to M$ is a projective cover iff it induces an isomorphism $\text{Hom}(M, L) \to \text{Hom}(P, L)$ for all irreducibles $L$.

First, $P \to M$ iff $\text{Hom}(M, L) \hookrightarrow \text{Hom}(P, L)$ for all irreducibles $L$. If $P \to M$, then by applying $\text{Hom}(-, L)$, which is left exact, we see that $\text{Hom}(M, L) \hookrightarrow \text{Hom}(P, L)$.

If the map $P \to M$ is not onto, then $\text{coker}(P \to M)$ is nonzero finitely generated, so it has irreducible quotient $L$. Then $M \to L$ is in the kernel of $\text{Hom}(M, L) \to \text{Hom}(P, L)$, so this map is not injective.

If $P \to M$ is a projective cover, and there exists $P \to L$ that doesn’t come from some $M \to L$, then $\text{ker}(P \to L) \to M$, so the surjection is not essential, a contradiction. If $P \to M$ is not a projective cover, then there exists $Q \leftarrow P$ with $Q \to M$. Then $P/Q$ has a simple quotient $L$, and the map $P \to P/Q \to L$ cannot come from a map $M \to L$: if it did, then $P \to M \to L$ should pull back to $Q \to M \to L$, but this is the zero map because it’s also the composition $Q \leftarrow P \to P/Q \to L$, which is zero. Hence $\text{Hom}(M, L) \to \text{Hom}(P, L)$ is not surjective.

By definition $0 \to P^{-i-1} \to P^{-i} \to C^{-i+1} \to 0$. If $P^{-i} \to C^{-i+1}$ is a projective cover, then $\text{Hom}(C^{-i+1}, L) \cong \text{Hom}(P^{-i}, L)$, iff $\text{Hom}(P^{-i}, L) \cong \text{Hom}(P^{-i+1}, L)$.

**Remark 10.11:** This generalizes to $\mathbb{Z}_{\geq 0}$-graded rings where $A_0$ is Artinian and $A_d$ is finitely generated over $A_0$. A common setting where this appears is an algebra $A$ over an algebraically closed field $k$ where $A_0$ is semisimple and $A_d$ is finite-dimensional over $k$. In this setting, there are still indecomposable projectives. In minimal resolutions, each term has finitely many generators in each degree. The graded irreducibles are concentrated in one degree (use that if $M$ is a graded $A$-module, then $M_{\geq k} := \bigoplus_{i \geq k} M_i \subset M$ is a $A$-submodule of $M$ for any $k \in \mathbb{Z}$). It follows that graded irreducible $A$-modules are annihilated by $A_{d+1}$ so they are just irreducible $A_d$-modules (up to a shift of grading).

If $A_0 = k$ is just a field, and for finitely generated (graded) $M$, we can consider its Poincare series $\sum_i \text{dim}(M_i)t^i \in \mathbb{Z}[[t]]$. More generally, if $A_0$ is semisimple then one can consider series $P_M := \sum_i[M_i]t^i \in \mathbb{Z}((t))^n$ where $n$ is the number of irreducibles for $A_0$ and $[M_i] \in K^0(A_0\text{-Mod}) \cong \mathbb{Z}^n$. The Cartan matrix $C$ now lies in $\text{GL}_n(\mathbb{Z}[[t]])$ instead of $\text{Mat}_n(\mathbb{Z})$ (it is clear that $C \in \text{Mat}_n(\mathbb{Z}[[t]])$ and $C(0) = \text{Id}$ since $A_0$ is semisimple, it then follows that $C \in \text{GL}_n(\mathbb{Z}[[t]])$). If $L_i$ has finite homological dimension, and $A$ is Noetherian then $C^{-1} \in \text{Mat}_n(\mathbb{Z}[[t]])$.

For example, if $A = k[x]$, considered as a graded algebra with $\deg x = 1$, then $n = 1, L_1 = k, P_1 = k[x]$, so $C = \sum_{i=0}^\infty t^i = \frac{1}{1-t}$, and $C^{-1} = 1 - t$. 

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