Lecture 11: Koszul Complexes

11.1 More on the Hattori-Stallings Dennis trace

Recall from Lemma 8.5 that the cokernel $R/[R, R] = C(R)$ receives a universal trace map $\tau(P, \varphi) \in C(R)$ where $P$ is a finitely generated projective and $\varphi \in \text{End}(P)$. In fact, if $R$ is 
Noetherian and of finite homological dimension, you can extend $\tau$ to $\tau(M, \varphi)$ where $M$ is any finitely generated module. To do so, choose a finite projective resolution
$0 \to P^{-n} \to P^{-n+1} \to \cdots \to P^0 \to M \to 0$ (which exists because $R$ has finite homological dimension). Then we can lift $\varphi$ to $\tilde{\varphi} \in \text{End}(P^*_M)$ and this will be unique up to homotopy. Define

$$\tau(M, \varphi) = \sum (-1)^i \tau(P^{-i}, \tilde{\varphi}^{-i})$$

which is well-defined because $M \mapsto P^*_M$ is a fully faithful functor to the homotopy category of complexes. Moreover, $\tau$ is additive on short exact sequences of modules.

**Corollary 11.1:** If $R$ is a finite-dimensional algebra of finite homological dimension over an algebraically closed
closed
field $k$, then $J(R) \subset [R, R]$.

**Proof.**

**Lemma 11.2:** For $M \in R\text{-Mod}$ and $\varphi \in \text{End}_R(M)$, we can find a $\varphi$-invariant Jordan-Holder series of $M$.

**Proof.** Consider $\varphi|_{\text{soc}(M)} : \text{soc}(M) \to \text{soc}(M)$, where $\text{soc}(M) = \bigoplus_i L_i^d_i$ is the socle of $M$. Then $\varphi$ induces an $R$-linear map $L_i^d_i \to L_i^d_i$ i.e. an element of $\text{End}_R(L_i^d_i) = \text{Mat}_d(k)$ (use Schur’s lemma) and this matrix has an
eigenvector, which generates a $\varphi$-invariant irreducible submodule in $M$. Then by inducting on the length of $M$, we get a $\varphi$-invariant Jordan-Holder series.

Thus, $\tau(M, \varphi) = \sum_i \tau(L_i, 1) = \sum_i \lambda_i \tau(L_i, 1)$ where $\lambda_i \in k$. It follows that the elements $\tau(L_i, 1) \in C(R)$ generate $C(R)$ as a vector space over $k$ (use Lemma 8.5 or Example 8.6). We conclude that $C(R)$ has dimension (over $k$) at most the number of irreducibles $L_i$. On the other hand, let $\bar{R} := R/J(R)$ and note that $C(R) \rightarrow C(\bar{R})$. It’s easy to see that $C(\bar{R}) = k^{a_j}$, so $C(R) \cong C(\bar{R})$ and $J(R) \subset [R, R]$.

**Question:** Is there a way to prove this without using the trace map?

11.2 Minimal resolutions and Koszul rings

Given a module $M$, how can we find its minimal resolution? For certain algebras called Koszul algebras, their minimal resolutions are called Koszul complexes. One great reference is [5, Section 2].

Let $A$ be a nonnegatively graded algebra over an algebraically closed field $k$ with $A_0$ semisimple. We will be interested in the case $A_0 = k$ so we can write $A = k \oplus A_{>0}$.

**Remark 11.3:** An elementary property of minimal resolutions for graded modules is that if $M = \bigoplus_{j \geq 0} M_j$, then

$P^{-i}$ must be concentrated in degrees $i$ and higher, since the projective cover $P \rightarrow M$ is an isomorphism in the

bottom degree (use that $A_0 = k$ is semisimple).

We will need the following technical lemma.

**Lemma 11.4:** Let $M$ be a finitely generated graded module over $A$. Then the following properties are equivalent:

(i) $M$ is generated by degree $i$ elements,

(ii) $M \otimes_A k$ is concentrated in degree $i$,

(iii) $\text{Hom}_A(M, k)$ is concentrated in degree $-i$.

**Proof.** Lemma follows from the Nakayama lemma together with the fact that

$$\text{Hom}_A(M, k) = \text{Hom}_{k_0}(M/A_{>0}M, k) = (M/A_{>0}M)^*.$$
**Definition 11.5:** We say that \( A \) is **Koszul** if \( P^{-i} \) is generated by degree \( i \) elements. Equivalently, \( \text{Tor}^A_k(k, k) \) (where each of the \( k \) are in degree 0) is concentrated in degree \( i \), which is equivalent to \( \text{Ext}^A_k(k, k) \) is concentrated in degree \( -i \) (use Lemma 11.2 above).

**Theorem 11.6:**

a) Koszul rings are **quadratic**, i.e. \( A = T(V)/\langle I \rangle \), where \( T(V) \) is the tensor algebra for a vector space \( V \) and \( I \) is a subspace of \( V \otimes V \).

b) If \( A \) is Koszul, then \( \text{Ext}^A_k(k, k) = A' \), where \( A' \) is the **dual quadratic algebra** \( T(V^*)/\langle I' \rangle \).

**Example 11.7:** Let \( A = T(V) \), so \( I = 0 \). Then the dual quadratic algebra is \( A' = T(V^*)/\langle V^* \otimes V^* \rangle = k \oplus V^* \).

Hence \( \text{Ext}_A(k, k) \) is only nonzero in degrees 0 and 1. \( k = T(V)/\langle V \rangle \) then has a free resolution in degrees 0 and 1.

**Example 11.8:** Let \( A = \text{Sym}(V) = T(V)/(\wedge^2 V) \). Then \( A' = T(V^*)/(\text{Sym}^2(V^*)) = \wedge^* V^* \).

**Definition 11.9:** The **dth Veronese subalgebra** \( A^{(d)} \) is \( \bigoplus_{n=0}^{\infty} A_{nd} \).

Let us mention the following theorem without a proof (see [3] for details).

**Theorem 11.10:** If \( A \) is a finitely generated commutative algebra, \( A^{(d)} \) is Koszul for large \( d \).

**Remark 11.11:** Using the approach of [6, Section 2] or [10] (see also Remark 12.2 below) one can easily prove (using Serre’s vanishing theorem) that for every \( m \in \mathbb{Z}_{>0} \) and large enough \( d \) (depending on \( m \)) the algebra \( A^{(d)} \) has the following property: \( P^{-i} \) is generated by degree \( i \) elements for \( i \leq m \). The statement of Theorem 11.10 is stronger, and the proof is more involved.

### 11.3 Koszul complexes

**Remark 11.12:** Assume \( A = T(V)/\langle I \rangle \) is quadratic. Then

\[
A_n = T^n(V)/\langle I \rangle_n = V^\otimes n \left( \sum_{i=0}^{n-2} V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2} \right).
\]

Define

\[
R_n := \bigcap_{i=0}^{n-2} V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2}
\]

to be the intersection rather than the sum. Then \( R_n = (A_n')^\ast \),

\[
R_n^\ast = V^\ast \otimes n \left( \sum_{i=0}^{n-2} (V^\ast)^{\otimes i} \otimes I^\perp \otimes (V^\ast)^{\otimes n-i-2} \right) = A_n'.
\]

**Definition 11.13:** The **Koszul complex**, denoted \( \mathbb{K}^\ast \), is a complex of free \( A \)-modules \( \cdots \rightarrow A \otimes_k R_2 \rightarrow A \otimes_k R_1 \rightarrow A \). As (graded) vector spaces, \( \mathbb{K}^\ast = \bigoplus_{n=0}^{\infty} \mathbb{K}_n^\ast \). The differential of \( \mathbb{K}_n^\ast \) is given by:

\[
\mathbb{K}^i_{n-i} = A_i \otimes R_{n-i} \hookrightarrow A_i \otimes V \otimes R_{n-i-1} \rightarrow A_{i+1} \otimes R_{n-i-1} = \mathbb{K}^{i+1}_{n-i}
\]

where the left map is induced by the natural embedding \( R_{n-i} \subset V \otimes R_{n-i-1} \) and the right map is induced by the multiplication \( A_i \otimes V \rightarrow A_{i+1} \).
Definition 11.14: Let $V$ be a vector space. A **distributive lattice** of subspaces of $V$ is a collection of subspaces satisfying

- For $Y$ in the lattice, $X \subseteq Y$ is also in the lattice
- For $X, Y$ in the lattice, $X + Y$ is also in the lattice
- For $X, Y, Z$ in the lattice, $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$ (distributivity).

Theorem 11.15 (Theorem 11.6 cont.):

a) Koszul rings are quadratic, i.e. $A = T(V)/\langle I \rangle$, where $T(V)$ is the tensor algebra for a vector space $V$ and $I$ is a subspace of $V \otimes V$.

b) If $A$ is Koszul, then $\text{Ext}_A^0(k, k) = A^!$, where $A^!$ is the dual quadratic algebra $T(V^*)/\langle I^! \rangle$.

c) Say $A$ is a quadratic algebra. It is Koszul iff $K$ is exact, i.e. $H^i(K) = 0$ for all $i \neq 0$, iff $K$ is the minimal resolution of the left module $k$.

d) Say $A$ is a quadratic algebra. It is Koszul iff for all $n$, the $n-1$ vector spaces $V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2}$, $i = 0, \ldots, n-2$, generate a distributive lattice of subspaces of $V^{\otimes n}$.

Lemma 11.16: A collection of vector subspaces in a vector space $W$ generate a distributive lattice iff there exists a basis of $W$ such that every subspace is spanned by a subset of the basis.

Proof. Clear.

Remark 11.17: The distributive property for the subspaces of $V^{\otimes n}$ described above is what implies the exactness of $K_n$. Moreover, the exactness of $K_{m,n}$, $m \leq n$, implies the distributive property for the subspaces of $V^{\otimes n}$.

For a collection $\mathcal{W} = (W; W_1, \ldots, W_n)$, where $W$ is a vector space and $W_1, \ldots, W_n \subset W$ are its subspaces let $K^{-l} = K^{-l}(\mathcal{W}) := \cap_{k=1}^{n-1} W_i/\langle W_{i+1} + \ldots + W_n \cap \left( \bigcap_{i=1}^{n-1} W_i \right) \rangle$, where $l = 0, 1, \ldots, n+1$.

For example, we have

$$K^{-n-1} = \bigcap_{i=1}^{n} W_i, \quad K^{-n} = \bigcap_{i=1}^{n-1} W_i, \quad K^{-n+1} = \bigcap_{i=1}^{n-2} W_i/\langle W_{n} \cap \left( \bigcap_{i=3}^{n} W_i \right) \rangle, \quad \ldots, \quad K^{-1} = W/\bigcap_{i=2}^{n} W_i, \quad K^0 = W/\bigcap_{i=1}^{n} W_i.$$

We have the natural maps $K^l \to K^{l+1}$ that make $K^* = K^*(\mathcal{W})$ into a complex.

Lemma 11.18: If $W_1, \ldots, W_n \subset W$ are proper subspaces and every proper subset of $\{W_1, \ldots, W_n\}$ generate a distributive lattice then $W_1, \ldots, W_n$ do the same iff $K^*(\mathcal{W})$ is exact.

Proof. It is clear that if $\{W_1, \ldots, W_n\}$ generate a distributive lattice then $K^*(\mathcal{W})$ is exact (for example, use Lemma 11.16). Assume now that $K^*(\mathcal{W})$ is exact. We prove the claim by the induction on $n$. We follow [4, Section 4.5].

We will use the following notations. Given a collection $U_1, \ldots, U_n \subset U$, say that a subspace $B \subset U$ is a splitting for $(U; U_1, \ldots, U_n)$ if there exists $C \subset U$ such that $B \oplus C = U$ and $(B \cap U_i) + (C \cap U_i) = U_i$. We will say that $(U; U_1, \ldots, U_n)$ is indecomposable if $U$ has no proper nonzero subspaces that split $(U; U_1, \ldots, U_n)$. The following easy facts will be extremely useful.

**Fact (1):** The subspace $U_1 \cap \ldots \cap U_i$ or $U_1 + \ldots + U_i$ is a splitting for $(U; U_1, \ldots, U_n)$ iff it is a splitting for $(U; U_{i+1}, \ldots, U_n)$.

Proof. Clear.

**Fact (2):** Assume that $(U_1 + \ldots + U_i) \cap (U_{i+1} \cap \ldots \cap U_j) = 0$ and $U_{i+1} \cap \ldots \cap U_j$ is a splitting for $(U; U_1 + \ldots + U_i, U_{i+1}, \ldots, U_n)$. Then $U_{i+1} \cap \ldots \cap U_j$ is a splitting for $(U; U_1, \ldots, U_n)$.
Proof. Let \((U_{i+1} \cap \ldots \cap U_j) \oplus B\) be a splitting for \((U; U_1 + \ldots + U_i, U_{i+1}, \ldots, U_n)\). Our goal is to check that it also gives a splitting for \((U; U_1, \ldots, U_n)\). From \((U_1 + \ldots + U_i) \cap (U_{i+1} \cap \ldots \cap U_j) = 0\) we conclude that \(U_1 + \ldots + U_i \subset B\) so \(U_1, \ldots, U_i \subset B\). It remains to check that \(U_k = (U_k \cap (U_{i+1} \cap \ldots \cap U_j)) + (U_k \cap B)\) for \(k = i + 1, \ldots, j\). This is clear since \(U_{i+1} \cap \ldots \cap U_j \subset U_k\).

\[\square\]

Fact (2'): Assume that \((U_1 \cap \ldots \cap U_i) \cap (U_{i+1} + \ldots + U_j) = 0\) and \(U_1 \cap \ldots \cap U_i\) is a splitting for \((U; U_{i+1} + \ldots + U_j, U_{j+1}, \ldots, U_n)\). Then \(U_i \cap \ldots \cap U_j\) is a splitting for \((U; U_1, \ldots, U_n)\).

Proof. Same proof as the one of Fact 2.

\[\square\]

Let us now return to the proof. Without losing the generality, we can assume that \(W = (W; W_1, \ldots, W_n)\) is indecomposable and all \(W_i\) are nonzero (and proper).

It then follows (use that by the inductive assumption, \(W_1 \cap W_2, W_3, \ldots, W_n \subset W, W_1, \ldots, W_{n-2}, W_{n-1} + W_n \subset W\) form distributive lattices and then apply Fact 1) that:

\[
W_1 \cap W_2 = 0, W_{n-1} + W_n = W.
\]

We can assume that \(n \geq 4\) (for \(n = 3\) the statement is clear, use exactness of \(K^*\)\((W)\)).

Assume that \(n = 4\). We have \(W_1 \cap W_3 \cap W_4 = 0 = W_2 \cap W_3 \cap W_4\) (use Fact 1). We also have

\[
(W_1 + W_2) \cap W_3 \cap W_4 = ((W_1 + W_2) \cap W_3) \cap ((W_1 + W_2) \cap W_4) = ((W_1 \cap W_3) + (W_2 \cap W_3)) \cap ((W_1 \cap W_4) + (W_2 \cap W_4)).
\]

We claim that the intersection \(((W_1 \cap W_3) + (W_2 \cap W_3)) \cap ((W_1 \cap W_4) + (W_2 \cap W_4))\) is zero. Indeed, if \(a + b = c + d\) for some \(a \in W_1 \cap W_3, b \in W_2 \cap W_3, c \in W_1 \cap W_4, d \in W_2 \cap W_4\) then \(a - c = d - b\) must lie in \(W_1 \cap W_2 = 0\) i.e. \(a = c \in W_1 \cap W_3 \cap W_4 = 0, d = b \in W_2 \cap W_3 \cap W_4 = 0\) so \(a = b = c = d = 0\). We conclude that \((W_1 + W_2) \cap W_3 \cap W_4 = 0\).

It then follows from Fact 2 that \(W_3 \cap W_4\) splits \((W; W_1, W_2, W_3, W_4)\) so we must have \(W_3 \cap W_4 = 0\) i.e. \(W = W_3 \oplus W_4\). It remains to note that \(W = W_3 \oplus W_4\) is splitting for \((W; W_1, W_2, W_3, W_4)\), and a contradiction finishes the argument. If \(n > 4\). The property \([2]\) implies that \((W; W_1, \ldots, W_n)\) remains acyclic after arbitrary transpositions of \(W_1, \ldots, W_{n-2}\) (by acyclic, we mean that the corresponding complex \(K^*\) is exact, it will be equal to zero in this case). So we may assume that for certain \(1 \leq i \leq n - 3\) one has \(A = W_i \cap \ldots \cap W_i \neq 0\) and each \(i + 1\)-tuple from \(W_1, \ldots, W_{n-3}\) intersects by zero. Put \(B = U_{i+1} + \ldots + U_{n-2}\). Then \((W; A; B; W_{n-1}, W_n)\) satisfies the assumptions of Lemma \[1.15\] (acyclicity follows from the fact that \(A \cap B = 0\) and \(W_{n-1} + W_n = W\) so (from \(n = 4\) case) we conclude that \(A; B; W_{n-1}, W_n \subset W\) generate a distributive lattice so \(A\) is a splitting for \((W; W_1, \ldots, W_n)\) by Fact 2'. Since \(A \neq 0\), we get a contradiction.

\[\square\]

Proof of Theorem \[1.15\]. If \(\text{Tor}_1(k, k)\) is concentrated in degree 1, then \(A_{\geq 1}\) is generated by degree 1 elements as an \(A\)-module (use the exact sequence \(0 \to A_{\geq 1} \to A \to k \to 0\) together with Nakayama). Hence, \(A\) is generated by degree 1 elements as a ring. Let \(V = A_1\) and write \(A = T(V)/I\). We have a map \(A \otimes V \to A\). Using that \(\text{Tor}_2(k, k)\) is concentrated in degree 2, we see that \(\ker(A \otimes V \to A)\) is generated by elements in \(A_1 \otimes V = V \otimes V\). These elements considered as elements of \(V \otimes V \subset T(V)\) generated the ideal \(\ker(T(V) \to A)\), so \(A\) is quadratic. Exactness of Koszul complex implies Koszul: If \(K_n\) is exact for \(n \geq 1\), then \(K\) is a free resolution of \(k\) as an \(A\)-module. So now we can use it to compute \(\text{Ext}_A^\bullet(k, k)\). Since \(R_n^\bullet \to R_{n-1}^\bullet\) and \(R_0^\bullet = A_1, \text{Ext}_A^n(k, k) = A_1^n\). You also have to check that this is compatible with multiplication, but after showing that, we can deduce that \(A\) is Koszul. To be continued next lecture.

\[\square\]
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