Lecture 11: Koszul Complexes

11.1 More on the Hattori-Stallings Dennis trace

Recall from Lemma 8.5 that the cocenter $R/[R,R] = C(R)$ receives a universal trace map $\tau(P,\varphi) \in C(R)$ where P is a finitely generated projective and $\varphi \in \mathrm{End}(P)$. In fact, if R is Noetherian and of finite homological dimension, you can extend τ to $\tau(M,\varphi)$ where M is any finitely generated module. To do so, choose a finite projective resolution $0 \to P^{-n} \to P^{-n+1} \to \cdots \to P^0 \to M \to 0$ (which exists because R has finite homological dimension). Then we can lift φ to $\tilde{\varphi} \in \mathrm{End}(P_M^\bullet)$ and this will be unique up to homotopy. Define

$$\tau(M,\varphi) = \sum_i (-1)^i \tau(P^{-i}, \tilde{\varphi}^{-i})$$

which is well-defined because $M \mapsto P_M^\bullet$ is a fully faithful functor to the homotopy category of complexes. Moreover, τ is additive on short exact sequences of modules.

Corollary 11.1: *If R is a finite-dimensional algebra of finite homological dimension over an algebraically closed field k, then $J(R) \subset [R,R]$.*

Proof.

Lemma 11.2: *For $M \in R$-Mod and $\varphi \in \mathrm{End}_R(M)$, we can find a φ-invariant Jordan-Holder series of M.*

Proof. Consider $\varphi|_{\mathrm{Soc}(M)} : \mathrm{Soc}(M) \to \mathrm{Soc}(M)$, where $\mathrm{Soc}(M) = \bigoplus_i L_i^{d_i}$ is the socle of M. Then φ induces an R-linear map $L_i^{d_i} \to L_i^{d_i}$ i.e. an element of $\mathrm{End}_R(L_i^{d_i}) = \mathrm{Mat}_{d_i}(k)$ (use Schur's lemma) and this matrix has an eigenvector, which generates a φ-invariant irreducible submodule in M. Then by inducting on the length of M, we get a φ-invariant Jordan-Holder series. □

Thus, $\tau(M,\varphi) = \sum_i \tau(L_i,\lambda_i) = \sum_i \lambda_i \tau(L_i, 1)$ where $\lambda_i \in k$. It follows that the elements $\tau(L_i,1) \in C(R)$ generate $C(R)$ as a vector space over k (use Lemma 8.5 or Example 8.6). We conclude that $C(R)$ has dimension (over k) at most the number of irreducibles L_i. On the other hand, let $\bar{R} := R/J(R)$ and note that $C(R) \twoheadrightarrow C(\bar{R})$. It's easy to see that $C(\bar{R}) = k^{\# L_i}$, so $C(R) \cong C(\bar{R})$ and $J(R) \subset [R,R]$. □

Question : Is there a way to prove this without using the trace map?

11.2 Minimal resolutions and Koszul rings

Given a module M, how can we find its minimal resolution? For certain algebras called Koszul algebras, their minimal resolutions are called Koszul complexes. One great reference is [5, Section 2].

Let A be a nonnegatively graded algebra over an algebraically closed field k with A_0 semisimple. We will be interested in the case $A_0 = k$ so we can write $A = k \oplus A_{>0}$.

Remark 11.3: An elementary property of minimal resolutions for graded modules is that if $M = \bigoplus_{i \geq 0} M_i$, then P^{-i} must be concentrated in degrees i and higher, since the projective cover $P \twoheadrightarrow M$ is an isomorphism in the bottom degree (use that $A_0 = k$ is semisimple).

We will need the following technical lemma.

Lemma 11.4: *Let M be a finitely generated graded module over A. Then the following properties are equivalent:*
(i) M is generated by degree i elements,
(ii) $M \otimes_A k$ is concentrated in degree i,
(iii) $\mathrm{Hom}_A(M,k)$ is concentrated in degree $-i$.

Proof. Lemma follows from the Nakayama lemma together with the fact that

$$\mathrm{Hom}_A(M,k) = \mathrm{Hom}_{A_0}(M/A_{>0}M, k) = (M/A_{>0}M)^*.$$

■ □

Definition 11.5: *We say that A is* ***Koszul*** *if P^{-i} is generated by degree i elements. Equivalently, $\operatorname{Tor}_i^A(k,k)$ (where each of the k are in degree 0) is concentrated in degree i, which is equivalent to $\operatorname{Ext}_A^i(k,k)$ is concentrated in degree $-i$ (use Lemma 11.2 above).*

Theorem 11.6:

a) *Koszul rings are* ***quadratic****, i.e. $A = T(V)/\langle I\rangle$, where $T(V)$ is the tensor algebra for a vector space V and I is a subspace of $V \otimes V$.*
b) *If A is Koszul, then $\operatorname{Ext}_A^\bullet(k,k) = A^!$, where $A^!$ is the* ***dual quadratic algebra*** *$T(V^*)/\langle I^\perp\rangle$.*

Example 11.7: Let $A = T(V)$, so $I = 0$. Then the dual quadratic algebra is $A^! = T(V^*)/\langle V^* \otimes V^*\rangle = k \oplus V^*$. Hence $\operatorname{Ext}_A(k,k)$ is only nonzero in degrees 0 and 1. $k = T(V)/\langle V\rangle$ then has a free resolution in degrees 0 and 1.

Example 11.8: Let $A = \operatorname{Sym}(V) = T(V)/\langle \wedge^2 V\rangle$. Then $A^! = T(V^*)/\langle \operatorname{Sym}^2(V^*)\rangle = \bigwedge^\bullet V^*$.

Definition 11.9: *The d**th Veronese subalgebra** $A^{(d)}$ is $\bigoplus_{n=0}^\infty A_{nd}$.*

Let us mention the following theorem without a proof (see [3] for details).

Theorem 11.10: *If A is a finitely generated commutative algebra, $A^{(d)}$ is Koszul for large d.*

Remark 11.11: Using the approach of [6, Section 2] or [10] (see also Remark 12.2 below) one can easily prove (using Serre's vanishing theorem) that for every $m \in \mathbb{Z}_{\geqslant 0}$ and large enough d (depending on m) the algebra $A^{(d)}$ has the following property: P^{-i} is generated by degree i elements for $i \leqslant m$. The statement of Theorem 11.10 is stronger, and the proof is more involved.

11.3 Koszul complexes

Remark 11.12: Assume $A = T(V)/\langle I\rangle$ is quadratic. Then

$$A_n = T^n(V)/\langle I\rangle_n = V^{\otimes n}\Big/\left(\sum_{i=0}^{n-2} V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2}\right).$$

Define

$$R_n := \bigcap_{i=0}^{n-2} V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2}$$

to be the intersection rather than the sum. Then $R_n = (A_n^!)^*$,

$$R_n^* = V^{*\otimes n}\Big/\left(\sum_{i=0}^{n-2} (V^*)^{\otimes i} \otimes I^\perp \otimes (V^*)^{\otimes n-i-2}\right) = A_n^!. \tag{1}$$

Definition 11.13: *The* ***Koszul complex****, denoted $\mathbb{K}^\bullet$, is a complex of free A-modules $\cdots \longrightarrow A \otimes_k R_2 \longrightarrow A \otimes_k R_1 \longrightarrow A$. As (graded) vector spaces, $\mathbb{K}^\bullet = \bigoplus_{n=0}^\infty \mathbb{K}_n^\bullet$. The differential of $\mathbb{K}_n^\bullet$ is given by:*

$$\mathbb{K}_n^{i-n} = A_i \otimes R_{n-i} \hookrightarrow A_i \otimes V \otimes R_{n-i-1} \longrightarrow A_{i+1} \otimes R_{n-i-1} = \mathbb{K}_n^{i+1-n}$$

where the left map is induced by the natural embedding $R_{n-i} \subset V \otimes R_{n-i-1}$ and the right map is induced by the multiplication $A_i \otimes V \longrightarrow A_{i+1}$.

Definition 11.14: *Let V be a vector space. A* ***distributive lattice*** *of subspaces of V is a collection of subspaces satisfying*

- *For Y in the lattice, $X \subset Y$ is also in the lattice*
- *For X, Y in the lattice, $X + Y$ is also in the lattice*
- *For X, Y, Z in the lattice, $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$ (distributivity).*

Theorem 11.15 (Theorem 11.6 cont.):

a) *Koszul rings are* ***quadratic****, i.e. $A = T(V)/\langle I \rangle$, where $T(V)$ is the tensor algebra for a vector space V and I is a subspace of $V \otimes V$.*

b) *If A is Koszul, then $\mathrm{Ext}^{\bullet}_A(k, k) = A^!$, where $A^!$ is the* ***dual quadratic algebra*** *$T(V^*)/\langle I^{\perp} \rangle$.*

c) *Say A is a quadratic algebra. It is Koszul iff $\mathbb{K}$ is exact, i.e. $H^i(\mathbb{K}) = 0$ for all $i \neq 0$, iff $\mathbb{K}$ is the minimal resolution of the left module k.*

d) *Say A is a quadratic algebra. It is Koszul iff for all n, the $n-1$ vector spaces $V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2}$, $i = 0, \ldots, n-2$, generate a distributive lattice of subspaces of $V^{\otimes n}$.*

Lemma 11.16: *A collection of vector subspaces in a vector space W generate a distributive lattice iff there exists a basis of W such that every subspace is spanned by a subset of the basis.*

Proof. Clear. □

Remark 11.17: The distributive property for the subspaces of $V^{\otimes n}$ described above is what implies the exactness of $\mathbb{K}_n$. Moreover, the exactness of $\mathbb{K}_m$, $m \leqslant n$, implies the distributive property for the subspaces of $V^{\otimes n}$.

For a collection $\mathcal{W} = (W; W_1, \ldots, W_n)$, where W is a vector space and $W_1, \ldots, W_n \subset W$ are its subspaces let $K^{-l} = K^{-l}(\mathcal{W}) := \bigcap_{i=1}^{l-1} W_i / \left((W_{l+1} + \ldots + W_n) \cap \left(\bigcap_{i=1}^{l-1} W_i \right) \right)$, where $l = 0, 1, \ldots, n+1$.

For example, we have

$$K^{-n-1} = \bigcap_{i=1}^{n} W_i,\ K^{-n} = \bigcap_{i=1}^{n-1} W_i,\ K^{-n+1} = \bigcap_{i=1}^{n-2} W_i \Big/ \left(W_n \cap \left(\bigcap_{i=3}^{n} W_i \right) \right), \ldots, K^{-1} = W \Big/ \sum_{i=2}^{n} W_i, K^0 = W \Big/ \sum_{i=1}^{n} W_i.$$

We have the natural maps $K^l \to K^{l+1}$ that make $K^{\bullet} = K^{\bullet}(\mathcal{W})$ into a complex.

Lemma 11.18: *If $W_1, \ldots, W_n \subset W$ are proper subspaces and every proper subset of $\{W_1, \ldots, W_n\}$ generate a distributive lattice then $W_1, \ldots, W_n$ do the same iff $K^{\bullet}(\mathcal{W})$ is exact.*

Proof. It is clear that if $\{W_1, \ldots, W_n\}$ generate a distributive lattice then $K^{\bullet}(\mathcal{W})$ is exact (for example, use Lemma 11.16).
Assume now that $K^{\bullet}(\mathcal{W})$ is exact. We prove the claim by the induction on n. We follow [4, Section 4.5].
We will use the following notations. Given a collection $U_1, \ldots, U_n \subset U$, say that a subspace $B \subset U$ is a splitting for $(U; U_1, \ldots, U_n)$ if there exists $C \subset U$ such that $B \oplus C = U$ and $(B \cap U_i) + (C \cap U_i) = U_i$. We will say that $(U; U_1, \ldots, U_n)$ is indecomposable if U has no proper nonzero subspaces that split $(U; U_1, \ldots, U_n)$. The following easy facts will be extremely useful.

Fact (1): *The subspace $U_1 \cap \ldots \cap U_i$ or $U_1 + \ldots + U_i$ is a splitting for $(U; U_1, \ldots, U_n)$ iff it is a splitting for $(U; U_{i+1}, \ldots, U_n)$.*

Proof. Clear. □

Fact (2): *Assume that $(U_1 + \ldots + U_i) \cap (U_{i+1} \cap \ldots \cap U_j) = 0$ and $U_{i+1} \cap \ldots \cap U_j$ is a splitting for $(U; U_1 + \ldots + U_i, U_{j+1}, \ldots, U_n)$. Then $U_{i+1} \cap \ldots \cap U_j$ is a splitting for $(U; U_1, \ldots, U_n)$.*

Proof. Let $(U_{i+1} \cap \ldots \cap U_j) \oplus B$ be a splitting for $(U; U_1 + \ldots + U_i, U_{j+1}, \ldots, U_n)$. Our goal is to check that it also gives a splitting for $(U; U_1, \ldots, U_n)$. From $(U_1 + \ldots + U_i) \cap (U_{i+1} \cap \ldots \cap U_j) = 0$ we conclude that $U_1 + \ldots + U_i \subset B$ so $U_1, \ldots, U_i \subset B$. It remains to check that $U_k = (U_k \cap (U_{i+1} \cap \ldots \cap U_j)) + (U_k \cap B)$ for $k = i+1, \ldots, j$. This is clear since $U_{i+1} \cap \ldots \cap U_j \subset U_k$. □

Fact (2'): *Assume that* $(U_1 \cap \ldots \cap U_i) \cap (U_{i+1} + \ldots + U_j) = 0$ *and* $U_1 \cap \ldots \cap U_i$ *is a splitting for* $(U; U_{i+1} + \ldots + U_j, U_{j+1}, \ldots, U_n)$. *Then* $U_1 \cap \ldots \cap U_i$ *is a splitting for* $(U; U_1, \ldots, U_n)$.

Proof. Same proof as the one of Fact 2. □

Let us now return to the proof. Without losing the generality, we can assume that $\mathcal{W} = (W; W_1, \ldots, W_n)$ is indecomposable and all W_i are nonzero (and proper).
It then follows (use that by the inductive assumption, $W_1 \cap W_2, W_3, \ldots, W_n \subset W$, $W_1, \ldots, W_{n-2}, W_{n-1} + W_n \subset W$ form distributive lattices and then apply Fact 1) that:

$$W_1 \cap W_2 = 0,\ W_{n-1} + W_n = W. \tag{2}$$

We can assume that $n \geqslant 4$ (for $n = 3$ the statement is clear, use exactness of $K^\bullet(\mathcal{W})$).
Assume that $n = 4$. We have $W_1 \cap W_3 \cap W_4 = 0 = W_2 \cap W_3 \cap W_4$ (use Fact 1). We also have

$$(W_1 + W_2) \cap W_3 \cap W_4 = ((W_1 + W_2) \cap W_3) \cap ((W_1 + W_2) \cap W_4) = ((W_1 \cap W_3) + (W_2 \cap W_3)) \cap ((W_1 \cap W_4) + (W_2 \cap W_4)).$$

We claim that the intersection $((W_1 \cap W_3) + (W_2 \cap W_3)) \cap ((W_1 \cap W_4) + (W_2 \cap W_4))$ is zero. Indeed, if $a + b = c + d$ for some $a \in W_1 \cap W_3$, $b \in W_2 \cap W_3$, $c \in W_1 \cap W_4$, $d \in W_2 \cap W_4$ then $a - c = d - b$ must lie in $W_1 \cap W_2 = 0$ i.e. $a = c \in W_1 \cap W_3 \cap W_4 = 0$, $d = b \in W_2 \cap W_3 \cap W_4 = 0$ so $a = b = c = d = 0$. We conclude that $(W_1 + W_2) \cap W_3 \cap W_4 = 0$. It then follows from Fact 2 that $W_3 \cap W_4$ splits $(W; W_1, W_2, W_3, W_4)$ so we must have $W_3 \cap W_4 = 0$ i.e. $W = W_3 \oplus W_4$. It remains to note that $W = W_3 \oplus W_4$ is splitting for $(W; W_1, W_2, W_3, W_4)$, and a contradiction finishes the argument.
If $n > 4$. The propperty (2) implies that $(W; W_1, \ldots, W_n)$ remains acyclic after arbitrary transpositions of $W_1, \ldots, W_{n-2}$ (by acyclic, we mean that the corresponding complex $K^\bullet$ is exact, it will be equal to zero in this case). So we may assume that for certain $1 \leqslant i \leqslant n-3$ one has $A = W_1 \cap \ldots \cap W_i \neq 0$ and each $i+1$-tuple from $W_1, \ldots, W_{n-2}$ intersects by zero. Put $B = U_{i+1} + \ldots + U_{n-2}$. Then $(W; A; B; W_{n-1}, W_n)$ satisfies the assumptions of Lemma 11.18 (acyclicity follows from the fact that $A \cap B = 0$ and $W_{n-1} + W_n = W$) so (from $n = 4$ case) we conclude that $A; B; W_{n-1}, W_n \subset W$ generate a distributive lattice so A is a splitting for $(W; W_1, \ldots, W_n)$ by Fact $2'$. Since $A \neq 0$, we get a contradiction.

□

Proof (of Theorem 11.15). If $\mathrm{Tor}_1(k,k)$ is concentrated in degree 1, then $A_{\geqslant 1}$ is generated by degree 1 elements as an A-module (use the exact sequence $0 \to A_{\geqslant 1} \to A \to k \to 0$ together with Nakayama). Hence, A is generated by degree 1 elements as a ring. Let $V = A_1$ and write $A = T(V)/I$. We have a map $A \otimes V \to A$. Using that $\mathrm{Tor}_2(k,k)$ is concentrated in degree 2, we see that $\ker(A \otimes V \to A)$ is generated by elements in $A_1 \otimes V = V \otimes V$. These elements considered as elements of $V \otimes V \subset T(V)$ generated the ideal $\ker(T(V) \twoheadrightarrow A)$, so A is quadratic.

Exactness of Koszul complex implies Koszul: If $\mathbb{K}_n$ is exact for $n \geqslant 1$, then $\mathbb{K}$ is a free resolution of k as an A-module. So now we can use it to compute $\mathrm{Ext}^\bullet_A(k,k)$. Since $R_n^* \xrightarrow{0} R_{n-1}^*$ and $R_n^* = A_n^!$, $\mathrm{Ext}^n_A(k,k) = A_n^!$. You also have to check that this is compatible with multiplication, but after showing that, we can deduce that A is Koszul. To be continued next lecture. □

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