Lecture 12: Koszul Rings Continued, Bar Complex

12.1 Finishing up Koszul rings

Proof (of Theorem 11.15, cont.) Subspaces $V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2} \subset V^{\otimes n}$, $i = 0, 1, \ldots, n-2$ generate a distributive lattice iff $\mathbb{K}_n^\bullet$ is exact: to see that it is enough to note that $\mathbb{K}_n^\bullet = K^\bullet(\mathcal{W})$ for

$$\mathcal{W} = (V^{\otimes n}; V^{\otimes n-2} \otimes I, V^{\otimes n-3} \otimes I \otimes V, \ldots, V^{\otimes n-i-1} \otimes I \otimes V^{\otimes i-1}, \ldots, I \otimes V^{\otimes n-2}).$$

Now the claim follows from Lemma 11.18 (using induction on n).

It's easy to see that $\mathbb{K}_n$ acyclic implies that $\mathbb{K}$ is a resolution for the trivial module, and $\mathrm{Tor}_i^A(k,k)$ is concentrated in degree i, so A is Koszul. In the other direction, suppose A is Koszul. We will inductively check acyclicity in the first d terms of the complex, which looks like $\cdots \to A \otimes I \to A \otimes V \to A$. If this complex is exact up to degree d, then the minimal space of generators for $\ker(A \otimes R_d \to A \otimes R_{d-1})$ is (some lift of) $\mathrm{Tor}_{d+1}^A(k,k)$. Because A is Koszul, this is in degree $d+1$, so it's a subspace in $A_1 \otimes R_d = V \otimes R_d$. It is the kernel of the multiplication map, so it must be R_{d+1}, so we're done. □

Remark 12.1: In commutative algebra, a "Koszul complex" often refers to a complex formed given a commutative ring R and n elements $x_1, \ldots, x_n \in R$. The last arrow in the complex is $R^{\oplus n} \to R$, sending $r_1, \ldots, r_n \mapsto \sum_{i=1}^n x_i r_i$. The Koszul complex for $\mathrm{Sym}(V)$ is an example of this.

Remark 12.2: We are now ready to give a sketch of the proof of the fact that for every $m \in \mathbb{Z}_{\geqslant 0}$, and large enough d, the algebra $A^{(d)}$ has the following property: P^{-i} is generated by degree i elements for $i \leqslant m$ (see Remark 11.11 above). So, our goal is to check that for every $n \in \mathbb{Z}_{\geqslant 0}$ the degree nth term of the Koszul complex for $A^{(d)}$ is exact for large enough d.
First of all, we can assume that A is generated by $A_1 = V$. Set $X := \mathrm{Proj}\, A$. We can assume that the natural morphism $X \hookrightarrow \mathbb{P}^N$ is a closed embedding. We have a natural (very ample) line bundle $\mathcal{O}_X(1)$ on X with $\Gamma(X, \mathcal{O}_X(1)) = A_1 = V$. Set $Y := X^n$, $\mathcal{L} := \mathcal{O}_X(1)^{\boxtimes n}$. For a closed $Z \subset Y$ we have $H^0(Y, \mathcal{L}) = V^{\otimes n}$ and denote by $Q_Z \subset V^{\otimes n}$ the kernel of $H^0(Y, \mathcal{L}) \to H^0(Z, \mathcal{L})$. Let $\Delta_i \subset X^n$ be the diagonal given by $x_i = x_{i+1}$. We have $Q_{\Delta_{i+1}} = V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2}$.
Let S^n be the (finite) set of closed subschemes of Y generated by $\{\Delta_i \mid i = 1, \ldots, n-1\}$ and $X^n, \varnothing$ via unions and (scheme-theoretic) intersections. Using Serre's vanishing theorem, we can assume that the statements of [6, Corollary 1.7] are satisfied for S^n. It then follows from [6, Lemma 2.1] that subspaces $V^{\otimes i} \otimes I \otimes V^{\otimes n-i-2}$ generate a distributive lattice of subspaces of $V^{\otimes n}$ so we are done by (the proof of) Theorem 11.15 (d).

Corollary 12.3: *The **Poincare series** of a graded algebra is*

$$P_A(t) = \sum d_n t^n, d_n = \dim A_n.$$

If A is Koszul, then $P_A(t) P_{A^!}(-t) = 1$.

Proof. This follows from the (graded) Euler characteristic of $\mathbb{K}$. If you look degree by degree, you can find that the Euler characteristic of $\mathbb{K}_n$ is the nth coefficient of $P_A(t)P_{A^!}(-t)$ (see (1)) so the total Euler characteristic of $\mathbb{K}$ is equal to $P_A(t)P_{A^!}(-t)$. Recall now that the Euler characteristic of $\mathbb{K}_n$ can also be computed as the alternating sum of dimensions of the cohomology of $\mathbb{K}_n$. It remains to note that $\mathbb{K}_n$ is exact for $n > 0$ and $\mathbb{K}_0 = k$ (sitting in degree 0). It follows that the total graded Euler characteristic of $\mathbb{K}$ is equal to 1. □

Example 12.4: Let $A = \bigwedge^n V$. Then $P_A(t) = (1+t)^n$. Likewise, $P_{\mathrm{Sym}(V)} = \frac{1}{(1-t)^n}$.

Proof (of Theorem 11.15, cont. again). Finally, we need to check that $A^! \simeq \mathrm{Ext}_A^\bullet(k,k)$ is an algebra isomorphism. First, we explain how to make $\mathrm{Ext}^\bullet$ into an algebra: $\mathrm{Ext}_A^\bullet(k,k) = H^*(\underline{\mathrm{Hom}}(P^\bullet, P^\bullet))$ for a projective resolution $P^\bullet$; $\underline{\mathrm{Hom}}$ is a DGA.
Here is how $A^!$ acts on $\mathbb{K}$: start with the action of $T(V^*)$ on $T(V)$ by contracting tensors $V^{*\otimes i} \times V^{\otimes n} \to V^{\otimes n-i}$. Restrict this to $V^{*\otimes i} \times R_n \to R_{n-i}$, which factors through $A_i^! \times R_n$. Recall that $\mathbb{K}^{-n} = A \otimes R_n$. Consider the map

$$(A \otimes R_n) \otimes A_i^! \to A \otimes R_{n-i} = \mathbb{K}^{-(n-i)}.$$

This is the $A^!$-action, and it commutes with the differential. Moreover, for $a \in A^!$, the composition $\mathbb{K} \xrightarrow{a} \mathbb{K} \to k$ represents the class of a. Hence, this is an algebra isomorphism. □

Remark 12.5: Let Proj_A be the projective graded A-modules. Then $A^!$ gives us an equivalence of derived categories

$$\mathcal{H}\wr(\mathrm{Proj}_A^{f.g.}) \simeq \mathcal{H}\wr(\mathrm{Proj}_{A^!}^{f.g.})$$

sending $M(1) \mapsto M[1](-1)$ where $M(1)_i = M_{i+1}$ and $[\cdot]$ is some homological stuff we won't discuss here. The idea is to use k as a generator for the derived category and consider the functor $F_k\colon M \to \mathrm{RHom}(k, M)$ which generalizes $F_P(M) = \mathrm{Hom}(P, M)$.

Remark 12.6: Let A_1, A_2 quadratic, $A_i = T(V_i)/I_i$. Then

$$A_1 \otimes A_2 = T(V_1 \oplus V_2)/I_1 \oplus I_2 \oplus \langle v_1 \otimes v_2 - v_2 \otimes v_1 \rangle$$

and

$$(A_1 \otimes A_2)^! = T(V_1^* \oplus V_2^*)/I_1^\perp \oplus I_2^\perp + \langle v_1 \otimes v_2 + v_2 \otimes v_1 \rangle,$$

the "super" (signed) tensor product.

Remark 12.7: If A is commutative and $I \supset \wedge^2(V)$, then $I^\perp \subset S^2V^*$. Then all the relations of $A^!$ will be relations between anticommutators and $A^!$ will be the enveloping algebra of a Lie superalgebra.

For more on Koszul rings, see [4] and [5].

12.2 Bar complex and Hochschild (co)homology

Definition 12.8: *Let A be any algebra over a field k. Then the* ***bar complex*** *of A is*

$$\cdots \to A \otimes_k A \otimes_k A \to A \otimes_k A \to A \to 0$$

where the last map is $a \otimes b \mapsto ab$ and in general

$$d\colon a_0 \otimes \cdots \otimes a_n \mapsto a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n + \cdots .$$

The RHS is also written as $a_0|a_1|\cdots|a_n$. Then $d^2 = 0$.

Lemma 12.9: *The bar complex is exact for any associative algebra.*

Proof. The map $h\colon a_0 \otimes \cdots \otimes a_n \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_n$ satisfies $dh + hd = \mathrm{id}$, so it is a chain homotopy. □

The bar complex is also a complex of A-bimodules. The left action is on a_0, and the right action is on a_n. A is the regular A-bimodule (i.e., $A \otimes_k A^{\mathrm{op}}$-module), and all the other terms are free, so the bar complex is a free resolution for A. This allows us to compute $\mathrm{Ext}^i_{A\otimes A^{\mathrm{op}}}(A, A)$ and $\mathrm{Tor}_i^{A\otimes A^{\mathrm{op}}}(A, A)$.

The bar complex also gives us a free resolution of every A-module by tensoring with M. The cohomology of the bar complex is $\mathrm{Tor}_i^A(A, M) = 0$ for $i > 0$.

Definition 12.10: *The* ***Hochschild homology*** *of A is the homology of the bar resolution. The* ***Hochschild cohomology*** *of A is the cohomology of $\mathrm{Hom}(\mathrm{Bar}, A)$, so the nth term is $A \otimes (A^{\otimes n-1})^*$. If A is graded, you can likewise define graded Hochschild cohomology.*

Remark 12.11: If A is augmented, you can use the reduced bar complex; let A_+ be the augmentation ideal, the reduced bar complex has terms $A \otimes_k A_+ \otimes_k \ldots \otimes_k A_+ \otimes_k A$. This allows you to compute $\mathrm{Ext}^i_A(k, k)$ and $\mathrm{Tor}_i^A(k, k)$, and indeed $A^!$ is in the bottom degree.