Lecture 13: Hochschild (co)Homology Continued, Central Simple Algebras

13.1 Deformations and Hochschild cohomology

From the definition of Hochschild (co)homology, we see that $\mathrm{HH}_0 = C(A) = A/[A,A]$ the cocenter and $\mathrm{HH}^0 = \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A,A) = Z(A)$ the center.

We also have a nice description for HH^1: the kernel of d is $\{\varphi\colon A \to A \mid \varphi(ab) = \varphi(a)b + a\varphi(b)\}$ and the image of d is $\{\varphi \mid \exists x \text{ s.t. } \varphi(a) = [a,x]\}$. So HH^1 is the derivations modulo the inner derivations, i.e., the outer derivations of A.

Lemma 13.1: $\mathrm{HH}^2(A)$ *is in bijection with isomorphism classes of first order deformations of* A.

Definition 13.2: *An* n***th order deformation*** *of* A *is an algebra* $\tilde{A}$ *free over* $k[t]/(t^{n+1})$ *and an isomorphism* $\tilde{A}/t\tilde{A} = A$. *A* ***formal deformation*** *of* A *is the same as above, but over* $k[[t]]$ *(and we need to use flatness instead of free), and a* ***polynomial deformation*** *of* A *is the one over* $k[t]$.

Proof. Suppose $\tilde{A}$ is a first order deformation of A and fix an isomorphism $\tilde{A} \simeq A \otimes_k (k[t]/(t^2))$. The multiplication μ on $\tilde{A}$ will correspond to a cocycle: it is determined by $\mu(a,b)$ for $a,b \in A$, and we must have $\mu(a,b) = ab$ modulo t, so we can say that $\mu(a,b) = ab + \varphi(a,b)t$ where $\varphi\colon A \otimes A \to A$. Then associativity of μ corresponds to φ being a cochain since we need

$$a\varphi(b,c) - \varphi(ab,c) + \varphi(a,bc) - \varphi(a,b)c = 0.$$

Given any cocycle, we can define a deformation of A by defining multiplication on $A \otimes k[t]/t^2$ to be $ab + \varphi(a,b)t$. An isomorphism of deformations $\widetilde{A_\varphi} \simeq_f \widetilde{A_\psi}$ is a map $f\colon \tilde{a} \mapsto \tilde{a} + tf(a)$ for $f\colon A \to A$, since again it only depends on the values it takes on A. f is an algebra homomorphism iff

$$(\psi - \varphi)(a,b) = af(b) - f(a)b,$$

that is, if $\psi - \varphi$ is a coboundary. □

Remark 13.3: Given an nth order deformation, the obstruction to extending it to an $n+1$th order deformation lies in $\mathrm{HH}^3(A)$; an expression in terms of the multiplication on $\tilde{A}$ must vanish in HH^3. Hence, if $\mathrm{HH}^3(A) = 0$, any deformation can be extended, and the set of all such extensions is in bijection with HH^2. However, this bijection is not canonical. Exercise: to get a canonical bijection, you also need the data of a torsor over HH^2.

Example 13.4: What is $\mathrm{HH}_\bullet(A)$ and $\mathrm{HH}^\bullet(A)$ for $A = k[x_1,\dots,x_n] = \mathrm{Sym}(V)$? For simplicity, assume char $k = 0$. We see we need to compute

$$\mathrm{Ext}^\bullet_{\mathrm{Sym}(V\oplus V)}(\mathrm{Sym}(V), \mathrm{Sym}(V))$$

and we already know how to do this: change coordinates using the Koszul complex to find that it's $\mathrm{Sym}(V) \otimes \bigwedge(V^*)$.
In particular, we remarked above that HH^1 is the outer derivations. For a commutative ring, there are no inner derivations, so $\mathrm{HH}^1(A)$ is exactly the derivations of $\mathrm{Sym}(V)$, which are

$$\left\{\sum_{i=1}^n p_i \partial_{x_i}\right\}, p_i \in k[x_1,\dots,x_n], \partial_{x_i}\colon P \to \frac{\partial P}{\partial x_i}.$$

Hence, $\mathrm{HH}^\bullet(A)$ is the polyvector fields on $V^* = \mathrm{Spec}(\mathrm{Sym}(V))$ and $\mathrm{HH}_\bullet(A) \simeq \mathrm{Sym}(V) \otimes \bigwedge V$, $\bigwedge V$ is in degree -1. These are the differential forms on V, Ω^i is in degree $-i$.

Remark 13.5: Hochschild-Kostant-Rosenberg generalized this to a smooth algebraic variety V. $\mathrm{HH}_\bullet$ and $\mathrm{HH}^\bullet$ carry more structure, related to differential geometry: the de Rham differential on forms corresponds to the Connes differential, which corresponds to cyclic cohomology. The latter uses the fact that the differential in the bar complex has cyclic symmetry.
The polyvector fields have a Schouten bracket, extending the commutator of vector fields $[v, w] = \mathrm{Lie}_v(w)$ (the Lie derivative). This generalizes to $\mathrm{HH}^\bullet(A)$, e.g. the obstruction in HH^3 for extending the 1st order deformation is $[h, h]$ where h is the deformation class.

13.2 Cobar complex and $A^!$

Let A be an augmented algebra, $A = k \cdot 1 \oplus A_+$, $A_+ = \bigoplus_{n\geqslant 1} A_n$. This induces a splitting of the bar resolution

$$A \otimes A_+ \otimes \cdots \otimes A_+ \otimes A \bigoplus \mathrm{span}(a_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes a_n).$$

This is because $d(\alpha \otimes 1 \otimes \beta) = d(\alpha) \otimes 1 \otimes \beta \pm \alpha \otimes 1 \otimes d(\beta) +$ stuff and you can check that the stuff is all like $\cdots a_{i-1} \otimes a_i \cdots - \cdots a_{i-1} \otimes a_i \cdots$ so it cancels. Therefore, both of the above are closed under the d-action. Hence, we can consider the reduced bar resolution and we can use it to compute graded $\mathrm{Ext}^\bullet_A(k, k)$ and show that it is $A^!$.

Define the graded dual of $M = \bigoplus M_i$ to be $M^* := \bigoplus M_i^*$; in this notation, the cobar complex is

$$A_+^* \to A_+^{*\otimes 2} \to \cdots$$

where the first is in degree $\leqslant -1$, the second is in degree $\leqslant -2$, and so on. Consider the degree $-i$ part in the ith term; it will equal $(V^*)^{\otimes i}$ where $V = A_1$, and

$$\mathrm{Ext}^i_A(k, k)_{-i} \simeq V^*/d()$$

where $d()$ is spanned by $d(a_1 \otimes a_2 \otimes \cdots \otimes b \otimes a_j \otimes \cdots \otimes a_i)$ where $a_k \in V^*$ and $b \in A_2^*$; this is

$$\pm d(a_1 \otimes a_2 \otimes \cdots \otimes db \otimes \cdots \otimes a_i).$$

So $d\colon A_2^* \to A_1^* \otimes A_1^*$, $A_2 = A_1 \otimes A_1/I$, and I is the space of degree 2 relations. $A_2^* = I^\perp \xhookrightarrow{d} V^* \otimes V^*$. So $V^*/d() \simeq A_i^!$, the quadratic dual to the quadratic part of A.

The cobar complex above is a DGA acting on the bar resolution of k. Hence, $A^! \simeq \bigoplus \mathrm{Ext}^i_A(k, k)_{-i}$ is an algebra isomorphism.

Note : For our next topic, we'll need that $H^*(G, M) = \mathrm{Ext}^\bullet_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ where G is a group (see Section 15.2 below).

13.3 Central simple algebras and Brauer group

We will look at simple Artinian rings R, so they are of the form $R = \mathrm{Mat}_n(D)$ for D a skew field. The center of D is a field k; then we say that R is a **central simple algebra** over k. We want to understand central simple algebras of finite dimension over a given field k.

Theorem 13.6:
a) *If A, B are two finite-dimensional central simple algebras over k, so is $A \otimes_k B$.*
b) *Consider the set of finite-dimensional central simple algebras over k modulo Morita equivalence. This set is in bijection with central division rings over k of finite dimension. With the operation $[A] + [B] := [A \otimes_k B]$, this set forms an abelian group, called the **Brauer group** of k.*

Lemma 13.7: *If A is a finite-dimensional central simple algebra over k, then $A_e := A \otimes_k A^{\mathrm{op}} \simeq \mathrm{End}_k(A)$.*

Proof. A is a simple algebra iff A is a simple A_e-module (A-bimodule). So $Z(A) = \mathrm{End}_{A_e}(A) \simeq k$ and A is finite-dimensional over k. Then by the density theorem, $A_e \twoheadrightarrow \mathrm{End}_k(A)$. If $d = \dim_k(A)$, then $\dim_k(A_e) =$

$\dim_k(\mathrm{End}(A)) = d^2$, so in fact this surjection is an isomorphism. □

Theorem 13.8 (Azumaya-Nakayama): *Suppose A is a central simple algebra over k and B is any algebra over k. Then two-sided ideals in $A \otimes_k B$ are in bijection with two-sided ideals in B.*

Proof. Our goal is to describe submodules of the $A_e \otimes_k B_e$-module $A \otimes_k B$. Consider $A \otimes_k B$ as an $A_e \otimes_k k$-module first. Then it's a simple module tensored with vector space. Hence A_e-submodules of it are of the form $A \otimes_k V, V \subset B$ a subspace (this follows from the classification of submodules in a semisimple module). But $A \otimes_k V$ is a $k \otimes_k B_e$-submodule iff V is a B_e-submodule of B, so in fact V must be a two-sided ideal of B. □

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