# Lecture 14: Brauer Group and Skolem-Noether Theorems



## 14.1 Definition and first properties of Brauer group

**Lemma 14.1:** *The center of a simple ring is a field.*

*Proof.* Saying that $A$ is a simple ring, i.e. it has no nontrivial proper two-sided ideals, is equivalent to saying that $A$ is simple as an $A \otimes_k A^{\text{op}}$-module. Then $\text{Hom}_{A\otimes_k A^{\text{op}}}(A, A) = Z(A)$ and by Schur's Lemma, it must be a division ring. It remains to note that every commutative division ring is a field. □

**Lemma 14.2:** *For $A, B$ two algebras over $k$, $Z(A \otimes_k B) = Z(A) \otimes_k Z(B)$.*

*Proof.* Suppose $x \in A \otimes B$ is central. We can write $x = \sum a_i \otimes b_i$ where the $a_i \in A$ are linearly independent and likewise for the $b_i \in B_i$. Then for all $a \in A$,

$$[x, a \otimes 1] = \sum [a, a_i] \otimes b_i = 0.$$

Since the $b_i$ are linearly independent, this implies the $a_i$ are all central. Likewise, $b_i \in Z(B)$. □

*Proof (of Theorem 13.6).* *a)* By Theorem 13.8, the tensor product $A \otimes_k B$ is a simple ring, and by the above lemmas its center is the field $Z(A) \otimes_k Z(B) = k$.

*b)* The tensor operation is well-defined up to Morita equivalence since $A \sim \text{Mat}_n(A)$ and

$$\text{Mat}_n(A) \otimes_k B = \text{Mat}_n(k) \otimes_k A \otimes_k B = \text{Mat}_n(A \otimes_k B).$$

The operation is obviously commutative and associative, has identity $k$, and inverse $-[A] = [A^{\text{op}}]$ since $[A \otimes_k A^{\text{op}}] = [\text{End}_k(A)] = [k]$.
To see that the set is in bijection with division rings over $k$ of finite dimension, note that Theorem 2.16 implies that any central simple algebra $A$ with center $k$ has the form $\text{Mat}_n(D)$ where $D$ is a skew field with center $k$. $D$ is unique because we can define $D$ as $\text{End}_A(L)^{\text{op}}$ where $L$ is the unique simple $A$-module.

□

**Example 14.3:** $\text{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ because there are exactly two finite-dimensional skew fields over $\mathbb{R}$, namely $\mathbb{R}$ and $\mathbb{H}$.

**Lemma 14.4:** *If $E/F$ is a field extension, then $A$ is a central simple algebra over $F$ iff $A \otimes_F E$ is a central simple algebra over $E$. More generally, if $B$ is an algebra over $E$, and $A$ is an algebra over $F$, then $A \otimes_F B$ is a central simple algebra over $E$ iff $A$ is a central simple algebra over $F$ and $B$ is a central simple algebra over $E$.*

*Proof.* Assume that $A/F$ and $B/E$ are central simple algebras. Then, by Theorem 13.8, $A \otimes_F B$ is a simple ring. Its center is (by lemma 14.2):

$$Z(A \otimes_F B) = Z(A) \otimes_F Z(B) = F \otimes_F E = E.$$

Assume now that $A \otimes_F B$ is a central simple algebra over $E$. Again from lemma 14.2 we know that $Z(A) \otimes_F Z(B) = Z(A \otimes_F B) = E$ so we must have $Z(A) = F$, $Z(B) = E$. It remains to note that if $A$ is not simple, then there exists a nonzero proper two-sided ideal $I \subset A$ but then $I \otimes_F B$ will be a nonzero proper two-sided ideal in $A \otimes_F B$. Contradiction finishes the proof. □

**Corollary 14.5:** *If $E/F$ is a field extension, it induces a group homomorphism called the **base change map***

$$\mathrm{Br}(F) \longrightarrow \mathrm{Br}(E), [A] \mapsto [A \otimes_F E].$$

*Proof.* It's a group homomorphism because

$$(A \otimes_F E) \otimes_E (E \otimes_F B) \cong E \otimes_F (A \otimes_F B).$$

□

**Example 14.6:** Algebraically closed fields have no finite skew field extensions, so if $k = \bar{k}$ then $\mathrm{Br}(k) = 0$. This implies that all central simple algebras over such $k$ are of the form $\mathrm{Mat}_d(k)$.

**Definition 14.7:** *Let $A$ be a central simple algebra over an arbitrary field $F$. The **degree of** $A$ is the $d$ such that*

$$A \otimes_F \bar{F} \cong \mathrm{Mat}_d(\bar{F}).$$

*Alternately, it is the $d$ such that $\dim_F(A) = d^2$.*

**Definition 14.8:** *The kernel of the base change map for an extension $E/F$ is denoted $\mathrm{Br}(E/F)$.*

**Definition 14.9:** *Let $A$ be a central simple algebra over $F$. We say an algebraic field extension $E/F$ **splits** $A$, or that $A$ **splits over** $E$, if $[A] \in \mathrm{Br}(E/F)$, i.e. $A \otimes_F E \cong \mathrm{Mat}_n(E)$.*

**Example 14.10:** Every central simple algebra $A$ over $F$ will split over $\bar{F}$.

**Corollary 14.11:** *Every central simple algebra $A$ over $F$ will split over a finite extension, namely the one generated by the matrix coefficients of the isomorphism $A \otimes_F \bar{F} \cong \mathrm{Mat}_n(\bar{F})$ (in some bases of $A$, $\mathrm{Mat}_n(F)$).*

## 14.2 Torsors and Galois forms

Classifying the central simple algebras of a fixed degree over a fixed field $F$ splitting over a fixed field extension of $E$ is a special case of **Galois forms** or the **Galois descent problem**. Here is an overview of the general procedure and the classification:

Assume that $E/F$ is Galois. Then consider the set $I$ of all $E$-linear isomorphism $A \otimes_F E \cong \mathrm{Mat}_n(E)$. $\mathrm{PGL}_n(E)$ acts on $\mathrm{Mat}_n(E)$ by conjugation; in fact, it is isomorphic to the group of automorphisms of $\mathrm{Mat}_n(E)$ (either a special case of the Theorem 14.14, see below, or a direct computation).

Hence, $\mathrm{PGL}_n(E)$ acts on $I$ by sending an isomorphism $A \otimes_F E \cong \mathrm{Mat}_n(E)$ to $A \otimes_F E \cong \mathrm{Mat}_n(E) \xrightarrow{\text{conj}} \mathrm{Mat}_n(E)$. It turns out that this action is *simply transitive*. On the other hand, we have an action of the Galois group $G = \mathrm{Gal}(E/F)$ on both $A \otimes_F E$ and on $\mathrm{Mat}_n(E)$, so it acts on $I$ by conjugation. These actions of $\mathrm{PGL}_n(E)$ and $G$ are compatible. This defines what we call a $\mathrm{PGL}_n(E)$**-torsor over** $G$.

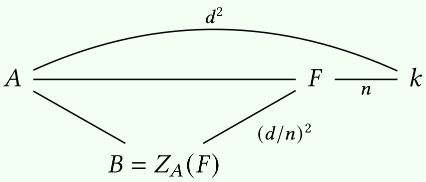
Hence, to every central simple algebra $A$ of degree $d$ split over $E$, we can assign a corresponding $\mathrm{PGL}_d(E)$-torsor over $G$, and it is not hard to see that this is a bijection. For example, the trivial torsor, where $I = \mathrm{PGL}_n(E)$, corresponds to $A \cong \mathrm{Mat}_n(F)$.

We will see in the next lecture that isomorphism classes of such torsors are classified by the nonabelian cohomology group $H^1(G, \mathrm{PGL}_n(E))$.

Moreover, this method generalizes to other algebraic objects depending on the choice of the base field, as long as "base change under field extension" makes sense: fix a reference object $S$, then the objects whose base change to $E$ are isomorphic to $S$ are in bijection with $\mathrm{Aut}(S)$-torsors over $G$.

## 14.3 Centralizer of a commutative subfield

**Lemma 14.12:** *If $k \subset F \subset A$ where $A$ is a central simple algebra over $k$, $\dim_k(A) = d^2$, $[F : k] = n$, and $B = Z_A(F)$, then $\dim_F(B) = \left(\frac{d}{n}\right)^2$ and $B$ is a central simple algebra over $F$ also. That is,*

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*Moreover, $[B] = [A \otimes_k F] \in \mathrm{Br}(F)$.*

*Proof.* $A \otimes_k F$ is a central simple algebra over $F$, and moreover it acts on $A$ by $a \otimes f : x \mapsto axf$. So $\mathrm{End}_{A \otimes_k F}(A) = Z_A(F) = B$ is also a central simple algebra and is Morita equivalent to $A \otimes_k F$ (recall that we have the natural identification $A \otimes_k A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_k(A)$ and $Z_A(F) \otimes_k A^{\mathrm{op}}$ identifies with $\mathrm{End}_F(A) \subset \mathrm{End}_k(A)$ so, by Lemma 14.4, $B = Z_A(F)$ is indeed a c.s.a. over $F$).
To find $\dim_F(B)$, notice that for any central simple algebra $C$ over $F$ and a $C$-module $M$ with $E = \mathrm{End}_C(M)$, we have

$$\dim_F(C)\dim_F(E) = \dim_F(M)^2.$$

Moreover, $C \otimes_F E \cong \mathrm{End}_F(M)$. This is because $C = \mathrm{Mat}_n(D)$, so $M = (D^n)^m$ for some $m$ and $E = \mathrm{Mat}_m(D^{\mathrm{op}})$. Then

$$C \otimes_F E = \mathrm{Mat}_{nm}(D \otimes D^{\mathrm{op}}) = \mathrm{Mat}_{nmd}(F) = \mathrm{End}_F(M)$$

where $d = \dim_F(D)$, and taking dimensions we get the desired identity.
Setting $C = A \otimes_k F, M = A, B = E$, we get

$$n^2 \dim_F(B) = d^2 \Rightarrow \dim_F(B) = \left(\frac{d}{n}\right)^2.$$

□

**Corollary 14.13:** *Let $A$ be a central simple algebra of degree $d$ over a field $k$. Then every subfield $F$ of $A$ has degree $\leqslant d$ over $k$. Moreover, field $F$ is a maximal commutative subalgebra of $A$ iff $[F : k] = d$.*

*Proof.* The fact that $[F : k] \leqslant d$ directly follows from Lemma 14.12.
If $F \subset A$ is maximal commutative, then $Z_A(F)$ must be equal to $F$ (indeed, otherwise there exists an element $x \in Z_A(F) \setminus F$ so $F[x]$ is a commutative subalgebra of $A$ that is bigger than $F$). So $Z_A(F) = F$ and the claim about the dimension of $F$ (over $k$) follows from Lemma 14.12. □

**Warning :** It may happen that $F \subset A$ is a maximal commutative *subfield* but not a maximal commutative *subalgebra* (take, for example, $A = \mathrm{Mat}_n(k)$ and $F = k$). If $A$ is a skew field, then these two properties do coincide.

## 14.4 Skolem-Noether

**Theorem 14.14 (Skolem-Noether):** *Let $A$ be a simple Artinian ring with center $k$ and $B$ a simple finite-dimensional $k$-algebra. Then any two $k$-linear homomorphisms $B \longrightarrow A$ are conjugate by an invertible element of $A$.*

This allows us to relate different embeddings of a given field in a central simple algebra.

*Proof.* Let $\varphi\colon B \to A, \psi\colon B \to A$ be two $k$-linear maps $B \to A$. These give $A$ two structures as an $(A, B)$-bimodule: $A_\varphi$ where

$$a \otimes b\colon x \mapsto ax\varphi(b)$$

and $A_\psi$ where

$$a \otimes b\colon x \mapsto ax\psi(b).$$

Since $A \otimes_k B^{\mathrm{op}}$ is simple (Theorem 13.8) and finitely generated as an $A$-module, it must be Artinian. So $A \otimes_k B^{\mathrm{op}}$ has only one simple module $L$, and any module $M$ finitely generated over $A$ will be isomorphic to $L^n$, $n < \infty$, and $n$ is determined by the isomorphism class of $M|_A$. Then $A_\varphi \cong A_\psi$. The isomorphism is given by right multiplication by some left invertible, hence invertible, element of $A$ that conjugates $\varphi$ into $\psi$. □

## 14.5 Artin-Wedderburn

**Theorem 14.15 (Artin-Wedderburn):** *There are no finite noncommutative skew fields. Hence, the Brauer group of a finite field is trivial.*

*Proof.* Suppose that $D$ is a noncommutative finite skew field with center $F = \mathbb{F}_q$. Let $E \subset D$ be a maximal commutative subfield. So by Corollary 14.13, $[E : F] = d$ where $d^2 = \dim_F(D)$. For $\alpha \in D$, $K = F(\alpha)$ will have degree $d'$ over $F$ with $d' \mid d$.
Then $E = \mathbb{F}_{q^d}$ and $K = \mathbb{F}_{q^{d'}}$. This implies that $K$ is isomorphic to a subfield in $E$ as an extension of $F$. This gives us two homomorphisms $E \to D$ and $K \to D$, so there exists an $x \in D^\times$ such that $xKx^{-1} \subset E$ by Theorem 14.14. $D^\times$ is a finite group and $E^\times \subset D^\times$ is a subgroup, and the following lemma implies that $E = D$.

**Lemma 14.16:** *Let $H \subset G$ be a subgroup in a finite group $G$. If every element in $G$ is conjugate to an element in $H$, then $H = G$.*

*Proof.* Let $\mathcal{C}$ be the set of conjugacy classes in $G$. For each conjugacy class $C \in \mathcal{C}$, we know $|C| = |G : Z_G(g)|$, $g \in C$, and $Z_G(g)$ is the centralizer of $g$. By assumption $C \cap H$ is nonempty for every conjugacy class, and we can bound

$$|C \cap H| \geqslant [H : C_H(g)] \geqslant \frac{[G : Z_G(g)]}{[G : H]} = \frac{|C|}{[G : H]}.$$

with equality when $C \cap H$ is single $H$-conjugacy class (first equality) and $Z_G(g) \subset H$ (second equality). In particular, if $g = 1$, we will always get a strict inequality. Then

$$|H| = \sum |C \cap H| > \frac{\sum |C|}{[G : H]} = \frac{|G|}{[G : H]},$$

contradiction. □

□

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