Lecture 16: Cohomological Description of the Brauer Group

More on the Brauer group.

16.1 Cross-product algebras and Galois extensions

Proposition 16.1: *Suppose E/F is a Galois extension. Then we have a bijection between central simple algebras over F with maximal commutative subfield (isomorphic to) E and cross-product extensions of $G = \mathrm{Gal}(E/F)$ by $E^\times$.*

Proof. The bijection will send a central simple algebra A with maximal commutative subfield E to $\tilde{G} = \mathrm{Nm}_{A^\times}(E)$, where Nm is for normalizer; this is a cross-product extension of G by $E^\times$. Since conjugating by an element of $\tilde{G}$ induces a Galois automorphism of E by definition, there is a homomorphism $\tilde{G} \to G$. Skolem-Noether implies that this is onto. The kernel of this homomorphism is the invertible elements of A that commute with E. Since $Z_A(E) = E$, the kernel must be $E^\times$ and we have an exact sequence $0 \to E^\times \to \tilde{G} \to G \to 0$, giving us a cross-product extension.

In the other direction, the bijection will take a cross-product extension, which corresponds to $c \in H^2(G, E^\times)$, to $A := G\#_c E$. First, we claim that A is a central simple algebra. First, it is simple. Notice that $E \otimes_F E \cong \prod_G E$ (by Galois theory) and A is a free rank 1 module over $E \otimes_F E$. Conjugation by an element $x_g \in A_c$, $x \neq 0$, will permute the copies of E and send E_h to $E_{h'}$. Therefore, for a nonzero ideal $I \subset A$, I must have a nonzero intersection with some E_g, hence it contains E_g, but then I contains all the E_g and $I = A$.

And $Z_A(E) = E$: if $x = (x_g) \in A$ with $x_g \neq 0$ and $g \neq 1$, we can pick $y \in E$ such that $g(y) \neq y$, in which case

$$(xy)_g = g(y)x \neq yx = (yx)_g.$$

Hence, $Z(A) \subset E$ and $Z(A) = E^G = F$.

Now we check these are inverse bijections. Start with $\tilde{G} = \tilde{G}_c$ and let $A = G\#_c E$. Then $\mathrm{Nm}_{A^\times}(E) = \tilde{G}$, since if $a \in A^\times$ normalizes E, then $ag^{-1} \in Z(E)$ for some $g \in G$, so $ag^{-1} \in E^\times$. Conversely, starting with A, mapping to a cocycle c, the map $(x_g) \mapsto \sum xg$ is a homomorphism. Then the map $G\#_c E \to A$ is injective because $G\#_c E$ is simple, and moreover, these have the same dimension over F, so the map is an isomorphism. □

Remark 16.2: While the above gives a transparent relation between central simple algebras and cross-products, some questions about this construction turn out to be quite hard. In particular, it's hard to determine whether a given cross-product algebra is a skew field or whether a given skew field is isomorphic to a cross-product algebra, see e.g. [2].

16.2 Maximal commutative subfields and splitting fields

Lemma 16.3: *Let E/F be a finite extension and A a central simple algebra over F. Then $[A] \in \mathrm{Br}(E/F)$ iff A is equivalent to an algebra A' containing E as a maximal subfield.*

Proof. Suppose that $E \subset A$ is a maximal subfield. Recall that A is isomorphic to a matrix algebra over some skew field D. It is enough to show that D splits over E. From the last lecture, we proved that for a central simple algebra D over F and a field $E \subset D$, $[D \otimes_F E] = [Z_D(E)]$. So if $Z_D(E) = E$, then $[Z_D(E)] = 0$ and D splits over E.
In the other direction, suppose that A splits over E and is represented by a skew field D. Write $A = \mathrm{Mat}_m(D)$ and consider the minimal n such that $\mathrm{Mat}_n(D) \supset E$ (as F-rings). Then we claim $A' := \mathrm{Mat}_n(D)$ contains E as a maximal subfield. Let $B = Z_{A'}(E)$. Then $[B] = [D \otimes_F E]$ (also from last time). Moreover, we claim that B cannot contain any nontrivial idempotents. Otherwise, $e\,\mathrm{Mat}_n(D)e$ would be a smaller central simple algebra in the same Brauer class containing E, as $x \mapsto ex$ would be a nonzero homomorphism $E \to e\,\mathrm{Mat}_n(D)e$. Hence B is a skew field. So if A splits over E, then $[B] = 0 \in \mathrm{Br}(E)$, and $B = E$ as it is a skew field. So A' contains $B = E$ as a maximal subfield. □

16.3 Proof of the theorem

Corollary 16.4: *Let E/F be a finite Galois extension. Then* $\mathrm{Br}(E/F) \cong H^2(\mathrm{Gal}(E/F), E^\times)$.

Proof. Now we know that there is a bijection between central simple algebras over F with maximal commutative subfield E and $H^2(G, E^\times)$. The Lemma 16.3 implies that every class A in $\mathrm{Br}(E/F)$ has a representative A' with maximal commutative subfield E, hence there is a map $\mathrm{Br}(E/F) \twoheadrightarrow H^2(G, E^\times)$ (currently just a map of sets, not a homomorphism). It is an injection (on sets), since two equivalent central simple algebras of the same degree are isomorphic: if $\mathrm{Mat}_n(D)$ and $\mathrm{Mat}_m(D)$ have the same dimension over their center, $m = n$. So we have a bijection between $\mathrm{Br}(E/F)$ and $H^2(G, E^\times)$.
We need to check that this is a group homomorphism. Let's rewrite the group structure on H^2 in terms of cross-products. Given $\tilde{G}_{c_1}$ and $\tilde{G}_{c_2}$, one can check that

$$\tilde{G}_{c_1c_2} \cong \tilde{G}_{c_1} \times_G \tilde{G}_{c_2}/(m, -m) \subset M \times M.$$

Now we want to check that

$$B := A_{c_1} \otimes_F A_{c_2} \sim A_{c_1c_2}.$$

But $B \supset E \otimes_F E = \prod_G E$. Let $e = 1_1 \in E \otimes_F E$. Then $eBe \cong A_{c_1c_2}$ and this represents the class $[A_{c_1}] + [A_{c_2}]$, so the group structures on both are compatible. □

16.4 Applications

Proof (of Theorem 14.15). Recall that we want to prove that there are no finite noncommutative skew fields. This is equivalent to proving that $\mathrm{Br}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is trivial, i.e. by the above, that $H^2(G, \mathbb{F}_{q^n}^\times) = 0$. The Galois group of this extension is $\mathbb{Z}/n\mathbb{Z}$. Pick a generator $\gamma \in \mathbb{Z}/n\mathbb{Z}$. For cyclic groups, we can use the following resolution of $\mathbb{Z}$ to compute $H^*(\mathbb{Z}/n\mathbb{Z}, M)$:

$$\cdots \to \mathbb{Z}[G] \xrightarrow{1+ +\cdots+\gamma^{n-1}} \mathbb{Z}[G] \xrightarrow{1-\gamma} \mathbb{Z}[G] \xrightarrow{1+\gamma+\cdots+\gamma^{n-1}} \mathbb{Z}[G] \xrightarrow{1-\gamma} \mathbb{Z}[G] \to \mathbb{Z}$$

where the leftmost arrow fits in the exact sequence because

$$(1-\gamma)\sum_{i=0}^{n-1} n_i\gamma^i = \sum_{i=0}^{n-1}(n_i - n_{i-1})\gamma^i = 0 \Leftrightarrow n_i = n_j \;\forall i, j.$$

The complex is 2-periodic, since

$$\left(\sum_{i=0}^{n-1}\gamma^i\right)\left(\sum_{i=0}^{n-1} n_i\gamma^i\right) = \left(\sum_{i=0}^{n-1} n_i\right)\left(\sum_{i=0}^{n-1}\gamma^i\right).$$

So $H^{2k}(\mathbb{Z}/n\mathbb{Z}, M) = M^G/\mathrm{Im}(\mathrm{Av})$, where $\mathrm{Av}\colon M \to M^G$ takes $m \mapsto \sum_{g\in G} g(m)$. Thus if E/F is a Galois extension with $G \cong \mathbb{Z}/n\mathbb{Z}$, which is our case, $\mathrm{Br}(E/F) = H^2(G, E^\times) = F^\times/\mathrm{Nm}(E^\times)$ where here Nm is the image of the norm map. But for $F = \mathbb{F}_q, E = \mathbb{F}_{q^n}$, $\mathrm{Nm}(x) = x^{(q^n-1)/q-1}$, so cyclicity of $E^\times$ implies that $\mathrm{Nm}\colon E^\times \twoheadrightarrow F^\times$ and $\mathrm{Br}(E/F) = 0$. □

Remark 16.5: We have shown that if E/F is a Galois extension with $G \cong \mathbb{Z}/n\mathbb{Z}$, then $\mathrm{Br}(E/F) = H^2(G, E^\times) = F^\times/\mathrm{Nm}(E^\times)$. The identification can be explicitly described as follows: recall that $\gamma \in G$ is a generator. Consider the "twisted polynomial algebra" $E\langle x;\gamma\rangle := \{\sum_i c_i x^i \mid c_i \in E\}$ with $xc = \gamma(c)x$ for $c \in E$. Pick $a \in F^\times$, the corresponding central simple algebra is $E\langle x;\gamma\rangle/(x^n - a)$ (such algebras are called *cyclic* algebras).

Example 16.6: $\mathrm{Br}(\mathbb{C}/\mathbb{R}) = \mathbb{R}^\times/\mathrm{Nm}(\mathbb{C}^\times) = \mathbb{Z}/2\mathbb{Z}$. It is easy to see that the element $[1] \in \mathbb{Z}/2\mathbb{Z}$ corresponds to the central simple algebra $\mathbb{H}$ of quaternions.

16.5 Index and period

Definition 16.7: *The* ***index*** *of an element in a Brauer group is the degree of its minimal representative. That is, the index of* $[\mathrm{Mat}_n(D)] = [D]$ *equals* d *if* D *is a skew field of dimension* d^2.

Definition 16.8: *The* ***period*** *of a central simple algebra* A *over* F *is the order of* $[A] \in \mathrm{Br}(F)$.

Lemma 16.9: *The period of an element in the Brauer group divides its index. In particular, the period is always finite, and* Br *is torsion.*

Proof. Let D be the skew field representative of this element, say it has degree d, with center F. We proved that D has a maximal subfield E such that E/F is separable in Proposition 15.2. Let K be a Galois extension of F containing E and $G = \mathrm{Gal}(E/F)$. Then $E = K^H$ for an index d subgroup $H \subset G$, $H = \mathrm{Gal}(K/E)$.
Now the lemma follows from the following fact about group cohomology: given a finite group G, $H \subset G$ of index d, and a G-module M, the kernel of $\mathrm{res}\colon H^i(G, M) \to H^i(H, M)$ is killed by d. This is because we can define a map $a\colon H^i(H, M) \to H^i(G, M)$ so that $a \circ \mathrm{res}$ is multiplication by d. For $i = 0$, this map sends $m \mapsto \sum_{g\in G/H} g(m)$, and in higher degrees, take an injective resolution of M over G, which will restrict to an injective resolution over H, then apply the above map to each term of the resolution.
Hence, the dth power of every element in the Brauer group vanishes. □

Not all integers arise as indexes of Brauer classes:

Lemma 16.10: *If* F *is a perfect characteristic* p *field, the Brauer group has no* p*-torsion.*

Proof. A separable finite extension E of F is also perfect. Hence $E^\times \to E^\times, x \mapsto x^p$ is an isomorphism, so it induces an automorphism $H^2(G, E^\times) \to H^2(G, E^\times)$. □

Finally, we give a cohomological description of $\mathrm{Br}(F)$ in terms of the absolute Galois group. We can describe by taking a limit of the $\mathrm{Br}(E/F)$, but we need to take into account that the absolute Galois group $G_F = \mathrm{Gal}(\bar{F}_{\mathrm{sep}}/F)$ (where $\bar{F}_{\mathrm{sep}}$ is the separable algebraic closure) is a profinite group. Hence, we need to consider continuous cohomology instead of normal cohomology, where all cocycles in the standard complex are required to be continuous. Then we can show that

$$H^2_{\text{cont}}(G_F, \bar{F}^\times_{\text{sep}}) = \varinjlim_E H^2(\text{Gal}(E/F), E^\times) = \varinjlim \text{Br}(E/F) = \text{Br}(F).$$