Lecture 18: Azumaya Algebras

18.1 Azumaya algebras

Let R be a commutative ring and A a ring over R that is finitely generated and projective (equivalently, locally free) as an R-module. Then the rank is a locally constant function on $\mathrm{Spec}(R)$. Assume this function is nowhere vanishing. It will also be occasionally convenient for us to assume that the rank is constant. Let us use the notation $A_S := S \otimes_R A$ for a homomorphism of rings $R \to S$.

Lemma 18.1: *For R, A as above the following are equivalent:*

a) *The map $A \otimes_k A^{\mathrm{op}} \to \mathrm{End}_R(A)$ is an isomorphism.*

b) *For every algebraically closed field k and a homomorphism $R \to k$, $A_k \cong \mathrm{Mat}_n(k)$.*

c) *For every maximal ideal $\mathfrak{m} \subset R$, let $k = R/\mathfrak{m}$; the ring A_k is a central simple algebra over k.*

Proof. We check that $a) \Rightarrow b) \Rightarrow c) \Rightarrow a)$.

$a) \Rightarrow b)$: since A is locally free, for every $R \to S$, $\mathrm{End}_S(A_S) = (\mathrm{End}_R(A))_S$. Thus property $a)$ is inherited by base change and A_k is a finite-dimensional central simple k-algebra. Hence it's isomorphic to $\mathrm{Mat}_n(k)$ when k is algebraically closed.

$b) \Rightarrow c)$: since $A_k \otimes_k \bar{k} \cong \mathrm{Mat}_n(\bar{k})$, A_k is a central simple algebra.

c) $\Rightarrow$ *a)*: If $\varphi: M \to N$ is a map of finitely generated modules over a commutative ring where N is projective that induces an isomorphism $M_k \to N_k$ for every $k = R/\mathfrak{m}$, then it is an isomorphism. This is because Nakayama's Lemma implies φ is surjective, so N projective implies $M \cong N \oplus \ker\varphi$; then another application of Nakayama's Lemma shows that $\ker\varphi = 0$. Applying this fact to $M = A \otimes_R A^{\text{op}}$, $N = \text{End}_R(A)$, we get the claim. □

Definition 18.2: *A ring A satisfying the equivalent conditions of the lemma is called an* ***Azumaya algebra*** *over R.*

Example 18.3: Let R be a Noetherian domain and A an R-algebra finitely generated as an R-module. Let F be the field of fractions of R, and suppose R_F is a central simple algebra over F. Then, for a finite localization $S = R_{(r)}$ $(r \in R)$, the ring A_S is an Azumaya algebra over S.

Example 18.4 (Differential operators in char p): Let k be a characteristic p field and $A = k\langle x, y\rangle/(yx - xy - 1)$ be the Weyl algebra. Then x^p, y^p are central in A since $\text{ad}(x^p) = \text{ad}(x)^p$ while $\text{ad}(x)^2(y) = [x, 1] = 0$, likewise for y. We claim that A is an Azumaya algebra over $R = k[x^p, y^p]$.
One can check that $\{x^m y^n \mid m, n \in \mathbb{Z}_{\geqslant 0}\}$ form a k-basis in A, so A is a free module over $k[x^p, y^p]$ with basis $x^m y^n$, $m, n \in \{0, \dots, p-1\}$. To check that it's an Azumaya algebra, it suffices to check this holds after an extension of scalars to $\bar{k}$, so WLOG we can assume that $k = \bar{k}$. Then by the Hilbert Nullstellensatz, maximal ideals in R are generated by $x^p - a, y^p - b$ for $a, b \in k$. Then

$$A_{a,b} := A/(x^p - a, y^p - b) \cong \text{Mat}_p(k) = \text{End}_k(k[x]/x^p)$$

where the isomorphism sends x to "multiplication by $x + \alpha$" and y to $\frac{\partial}{\partial x} + \beta$, $\alpha^p = a$, $\beta^p = b$.

Example 18.5 (Quantum torus): Let $A = \mathbb{C}\langle z, z^{-1}, t, t^{-1}\rangle/(zt = qtz)$ for $q \in \mathbb{C}^\times$ fixed constant. If q is a primitive order l root of unity, then A is an Azumaya algebra over $\mathbb{C}[z^l, z^{-l}, t^l, t^{-l}]$; the proof is similar to the previous example.

Azumaya algebras allow us to define the notion of a Brauer group over a ring. In particular, if A, B are Azumaya algebras over R, then so is $A \otimes_R B$.

Definition 18.6: *The* ***Brauer group*** *of a ring R is the set of Morita equivalence classes of Azumaya algebras over R;* $[A] + [B] = [A \otimes_R B]$, $-[A] = [A^{\text{op}}]$.

For a homomorphism $R \to S$, we have a base change homomorphism $\text{Br}(R) \to \text{Br}(S)$, given by $A \mapsto A_S$; this is a homomorphism since A_S is Azumaya over S and $(A \otimes_R B)_S \cong A_S \otimes_S B_S$.

Remark 18.7: $[A] = 0$ iff $A \cong \text{End}_R(M)$ where M is a finitely generated projective constant rank module over R. This does not necessarily imply that $A \cong \text{Mat}_n(R)$ as in the field case.

18.2 Cohomological description of the Brauer group over a ring - preliminary discussion

Recall that for a central simple algebra A over a field F, we proved that

a) A is split over some algebraic extension of F.

b) We can choose such an extension to be separable.

c) For a fixed splitting Galois extension E/F, the action of $G = \text{Gal}(E/F)$ on $A \otimes_F E \cong \text{Mat}_n(E)$ leads to the cohomological description of the Brauer group.

These statements can be generalized to a Noetherian commutative ring R.

18.3 Faithfully flat ring homomorphisms and faithfully flat descent

Let A be an Azumaya algebra over a Noetherian commutative ring R. The obvious analog of point 1 is some S with a map $R \to S$ such that A_S splits. However, in the ring setting, we can lose information from the base change map; for example, $R = S_1 \times S_2 \to S = S_1$. So we need an additional condition on S.

One that works well is that S is **faithfully flat** over R.

Definition 18.8: *A ring S is **flat** over R if the functor $M \mapsto M_S$, R-Mod $\to$ S-Mod is exact. It is **faithfully flat** over R if it is conservative, i.e., $M \to N$ is an isomorphism iff $M_S \to N_S$ is an isomorphism.*

Remark 18.9: If S is flat, the conservativity condition is equivalent to $M_S \neq 0$ for $M \neq 0$.

Definition 18.10: *Let M be an R-module. The **descent data** for M is a module $N = M_S$ over S, and an isomorphism ι between the base changes of N to $S \otimes_R S$ such that the three base changes to $S \otimes_R S \otimes_R S$ form a commutative diagram.*

Proposition 18.11: *If $R \to S$ is faithfully flat, the functor sending M to its descent data is an equivalence.*

Remark 18.12: There is a parallel algebraic geometry statement. Notice that $S_1 \otimes_R S_2$ is the coproduct in the category of commutative rings, and $\mathrm{Aff} = \mathrm{Comm}^{\mathrm{op}}$ (affine schemes) where R corresponds to $\mathrm{Spec}(R)$, so $\mathrm{Spec}(S_1 \otimes_R S_2) = \mathrm{Spec}(S_1) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S_2)$, the fiber product.
The descent data is parallel to how to define a vector bundle or sheaf on X by gluing the corresponding data for an open covering $X = \bigcup U_i$. Replace $\mathrm{Spec}(R)$ by X, $\mathrm{Spec}(S)$ by $U = \bigsqcup U_i$, and $S \otimes_R S$ is replaced by the disjoint union of $U_i \cap U_j$. The compatibility condition for base changes to $S \otimes_R S \otimes_R S$ correspond to checking that the data for $U_i \cap U_j \cap U_k$ makes sense.

Noetherian rings have a faithful flatness criterion.

Definition 18.13: *A homomorphism of commutative rings $R \to S$ is **formally smooth** if for every commutative ring $T = \tilde{T}/I$ with $I^2 = 0$ and compatible maps $R \to \tilde{T}$, $S \to T$, you can lift to $S \to \tilde{T}$.*

Example 18.14: Suppose R, S are finitely generated k-algebras with k a field, $T = k$, $\tilde{T} = k[t]/t^2$. Then $I = (t)$. We have maps $R \to k[t]/t^2$ and $S \to k$, and so also a map $R \to k$, so we have maximal ideals $\mathfrak{m}_S \subset S, \mathfrak{m}_R \subset R$ with residue field k. Extending a map $R \to k$ to a homomorphism to $k[t]/t^2$ is equivalent to specifying a vector in $(\mathfrak{m}_R/\mathfrak{m}_R^2)^*$, i.e. the **tangent vector** to $\mathrm{Spec}(R)$ at the corresponding point.
So formal smoothness implies that the map on tangent spaces induced by $R \to S$ is onto, a condition appearing in the definition of submersion in differential geometry.

Now suppose that R is Noetherian and S is finitely generated over S. If S is formally smooth over R, then it is flat over R. Moreover, if the map on k-points $\mathrm{Hom}(S, k) \to \mathrm{Hom}(R, k)$ is onto for every algebraically closed field k, then S is faithfully flat.

18.4 Universal splitting

Let A be an Azumaya algebra over R of constant rank d^2. Then we can construct an example of a faithfully flat ring S splitting R.

Theorem 18.15:

a) *Consider the functor F sending a commutative R-ring S to the set of isomorphisms $A_S \cong \mathrm{Mat}_d(S)$. This functor is representable and is represented by a ring S_{univ} finitely generated over R.*
b) *S_{univ} is formally smooth over R. If R is Noetherian, it is faithfully flat over R.*

Proof. Recall that A is a projective module over R (of rank d^2). So $A \cong e(R^N)$ as an R-module where $e \in \mathrm{Mat}_N(R)$ is an idempotent.
First consider the functor sending S to the S-module isomorphisms $A_S \cong S^{d^2}$. If $A \cong e(R^N)$; then such an isomorphism is equivalent to producing two matrices $i \in \mathrm{Mat}_{d^2,N}(S)$, $j \in \mathrm{Mat}_{N,d^2}(S)$ such that $ij = I_{d^2}$ and $ji = e$. These are degree 2 equations in the entries of i, j, while the requirement that the isomorphism is compatible with the algebra structure on $\mathrm{Mat}_d(S)$ is another collection of degree 2 equations on the matrix entries of i. So we can define S_{univ} as the quotient of the polynomial ring in $2Nd^2$ variables over S by the ideal generated by these degree 2 equations.
To check that S_{univ} is formally smooth over R, we show that if $A_T \cong \mathrm{Mat}_n(T)$, then $A_{\tilde{T}} \cong \mathrm{Mat}_n(\tilde{T})$ where $T = \tilde{T}/I, I^2 = 0$, since that's what it means to be able to lift to a map $S_{\mathrm{univ}} \to \tilde{T}$. Consider a rank 1 idempotent $e \in \mathrm{Mat}_n(T)$ (without loosing the generality we can assume that $e = e_{11}$). We will use the same notation for the corresponding element on A_T. So A_T maps isomorphically to $\mathrm{End}_T(A_T e)$. We can lift e to $\tilde{e} \in A_{\tilde{T}}$ such that $\tilde{e} \bmod I = e$. Then

$$A_{\tilde{T}} \to \mathrm{End}_{\tilde{T}}(A_{\tilde{T}} e)$$

is a map of free $\tilde{T}$-modules of rank d^2 that is an isomorphism modulo I, hence an isomorphism.
Hence, S_{univ} is flat over R when R is Noetherian. To check it is faithful, we need to check that $A_k \cong \mathrm{Mat}_d(k)$ for every algebraically closed field k, but this is one of the properties of Azumaya algebras. □

18.5 Rewriting cochain complex for $H^*(G, E^\times)$

Let's rewrite the complex used to compute $H^*(G, E^\times)$ for a finite Galois field extension E/F, $G = \mathrm{Gal}(E/F)$ in a way that can be generalized to Noetherian commutative rings. Recall that the nth term is $C^n = \mathrm{Map}(G^n, E^\times) = (\prod_{G^n} E)^\times$. From Galois theory, $\prod_G E \cong E \otimes_F E$ (this is an isomorphism of algebras). By induction, $\prod_{G^n} E = E \otimes_F E \otimes_F \cdots \otimes_F E$ where there are $n+1$ factors in the RHS. Thus

$$C^n = (E \otimes_F \cdots \otimes_F E)^\times$$

where there are $n+1$ factors in the RHS.

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