Lecture 19: Brauer Group of a Ring Continued, Localization

19.1 Amitsur cohomology

Let $F\colon$ Comm $\to$ Ab be a functor. We can generalize the complex from the previous lecture to F, though we will mostly use $R \mapsto \mathbb{G}_m(R) = R^\times$. Given a homomorphism $R \to S$ we can form the **Amitsur complex** as follows:

Write $S_R^{\otimes n} = S \otimes_R \cdots \otimes_R S$ with n factors in the RHS. Set

$$C^n := F(S_R^{\otimes n+1}),\ d_n := \sum_{k=0}^{n+1} (-1)^k F(i_k)\colon C^n \to C^{n+1}$$

where $i_k\colon S_R^{\otimes n+1} \to S_R^{\otimes n+2}$ is the insertion map that puts a 1 in the kth place, i.e.

$$s_0 \otimes \cdots \otimes s_n \mapsto s_0 \otimes \cdots \otimes s_{k-1} \otimes 1 \otimes \cdots \otimes s_n.$$

We denote its cohomology by $H^i_{S/R}(F)$.

Example 19.1: Let $R = F, S = E$ with E/F a finite Galois extension and let the functor be $\mathbb{G}_m$. Recall that there is an isomorphism

$$(E_F^{\otimes n+1})^\times \xrightarrow{\sim} \left(\prod_{G^n} E\right)^\times = \mathrm{Map}(G^n, E^\times).$$

Choosing the isomorphism amounts to defining pairwise distinct homomorphisms

$$h_{g_1\cdots g_n}(x_0 \otimes x_1 \otimes \cdots \otimes x_n) = x_0 g_1(x_1) g_1 g_2(x_2) \cdots g_1 \cdots g_n(x_n)$$

where $h_{g_1\cdots g_n} : E_F^{\otimes n+1} \to E$. This commutes with i_k since if you let $x_k = 1$, you skip the $(k+1)$th factor and you get

$$h_{g_1\cdots g_n}(x_0 \otimes x_1 \otimes \cdots \otimes x_{k-1} \otimes 1 \otimes x_{k+1} \otimes \cdots \otimes x_n) = h_{g_1,\ldots,g_{i-2},g_{i-1}g_i,g_{i+1},\ldots,g_n}(x_0 \otimes \cdots \otimes x_n).$$

Hence the Amitsur complex is the standard complex computing $H^*(G, E^\times)$.

Remark 19.2: The algebraic geometry interpretation: Since $\mathrm{Comm}^{\mathrm{op}} = \mathrm{Aff}$, F can also be interpreted as a contravariant functor $\mathrm{Aff} \to \mathrm{Ab}$. Then $S \to R$ corresponds to $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$; consider the analogous construction where you replace an affine scheme by a topological space, so we can instead consider morphisms $U \to X$ where U is the disjoint union $\bigsqcup U_i$ of open subsets in an affine covering of X. If the assignment of an abelian group to each $U_i \to X$ comes from a sheaf $\mathcal{F}$ on X, we recover the Cech complex for $H^*(X, \mathcal{F})$.

19.2 Relationship between Brauer group and Amitsur cohomology

We sketch how to correspond Azumaya algebras with a class in the second cohomology. Let A be an Azumaya algebra over R and choose an isomorphism $A_S \cong \mathrm{Mat}_n(S)$. Then we have two isomorphisms $A_{S_R^{\otimes 2}} \cong \mathrm{Mat}_n(S_R^{\otimes 2})$, and again, their ratio will be an Amitsur 1-cocycle c with nonabelian coefficients that is independent of the choice of isomorphism up to scaling. Hence it gives an element in $H^1_{S/R}(\mathrm{PGL}_n)$, where PGL_n is the functor $R \mapsto \mathrm{PGL}_n(R)$ (again, these will be nonabelian groups). Notice that $\mathrm{PGL}_n(R) = \mathrm{Aut}(\mathrm{Mat}_n(R))$ is an algebraic group and the homomorphism $\mathrm{GL}_n(R)/R^\times \to \mathrm{PGL}_n(R)$ may not be surjective (unlike in the field case).

Let's just assume that we can lift c to GL_n, e.g. the map $\mathrm{GL}_n(S \otimes_R S) \to \mathrm{PGL}_n(S \otimes_R S)$ is surjective, so c lifts to $\tilde{c} \in \mathrm{GL}_n(S \otimes_R S)$. Then we can get a cocycle in $H^2_{S/R}(\mathbb{G}_m)$ by the same procedure as in the field case: consider the differential of $\tilde{c}$, which takes values in $E^\times$, giving the desired cocycle.

Remark 19.3: In fact, one can find a faithfully flat S for which a lift $\tilde{c}$ exists, but the proof is beyond the scope of the lecture. Then you can define $H^i_{\mathrm{fl},A}(R, \mathbb{G}_m)$ (A for Amitsur) as $\mathrm{colim}_S H^i_{S/R}$ where the colimit is over all faithfully flat S. Restricting to étale S, you get $H^i_{\text{ét},A}(R, \mathbb{G}_m)$, and this coincides with the étale cohomology of $\mathrm{Spec}(R)$.
We have injective maps from $\mathrm{Br}(R)$ into $H^2_{\mathrm{fl},A}(R, \mathbb{G}_m)$ and $H^2_{\text{ét},A}(R, \mathbb{G}_m)$.

19.3 Final remarks on Brauer group

First, we describe how to generalize separable splittings to rings. It turns out that for an Azumaya algebra A over R, we can always find an étale, faithfully flat homomorphism $R \to S$ such that A_S splits.

Definition 19.4: *A ring homomorphism $R \to S$ is **étale** if for every commutative ring $T = \tilde{T}/I$ with $I^2 = 0$ and compatible maps $R \to \tilde{T}$, $S \to T$, there exists a unique compatible map $S \to \tilde{T}$.*

Exercise : A finite field extension is étale iff it is separable.

Theorem 19.5: *Let R be a (formally) smooth finitely generated commutative domain over an algebraically closed field and $F = \mathrm{Frac}(R)$. Then $\mathrm{Br}(R) \hookrightarrow \mathrm{Br}(F)$.*

Proof (Sketch). The proof involves an object called the **Brauer-Severi variety** (to be denoted by B). We need the notion of a line bundle (a locally free coherent sheaf of rank 1) and the fact that for a smooth variety X over a field and $U \subset X$ an open subvariety, every line bundle on U can be extended to one on X. This follows from the correspondence between line bundles and divisors and the fact that the closure of a divisor on U is a divisor on X. We also need the concept of an algebraic group action on an algebraic variety and the quotient by such an action. Let A be an Azumaya algebra on $X = \mathrm{Spec}(R)$ and $S = S_{\mathrm{univ}}$ be the universal splitting ring. Then $G = \mathrm{PGL}_n$ acts on $Y = \mathrm{Spec}(S)$ so that $Y/G \cong X$. Recall that G also acts on $\mathbb{P}^{n-1}$. Set

$$B := (\mathbb{P}^{n-1} \times Y)/G.$$

Thus $B \to X$ and every geometric fiber of this map is isomorphic to $\mathbb{P}^{n-1}$. Then one can check that A is split iff there exists a line bundle L on B whose restriction to a geometric fiber is isomorphic to the line bundle $O(1)$ on $\mathbb{P}^{n-1}$. If A_F splits then there exists a nonempty open $U \subset X$ such that A_U splits, so A splits. □

19.4 Localization

Let R be a ring and S a multiplicatively closed subset, i.e. $1 \in S$ and $a, b \in S \implies ab \in S$.

Definition 19.6: *The* ***localization*** R_S *of* R *at* S *is the universal ring receiving a homomorphism from* R *sending* S *to invertible elements. That is,*

$$\mathrm{Hom}(R_S, T) = \{f : R \to T \mid f(s) \text{ is invertible } \forall s \in S\}.$$

The Yoneda Lemma shows that R_S is unique up to unique isomorphism if it exists.

Lemma 19.7: $R_S = R\langle t_s \rangle_{s \in S}/(t_s s = s t_s = 1)$.

19.5 Ore conditions

Unlike in the commutative ring case, it is hard to say much about R_S from this construction; for example, we don't even know if R_S is the zero ring. We can impose additional conditions on S to give R_S an explicit description.

Definition 19.8: *Let* $S \subset R$ *be a multiplicative subset. The* ***(right) Ore conditions*** *are*

- *(O1) For all* $a \in R, s \in S$, *then* $aS \cap sR \neq \varnothing$.
- *(O2) For all* $a \in R, s \in S$, *if* $sa = 0$, *then there exists* $t \in S$ *such that* $at = 0$.

If S *satisfies O1, it is called a* ***right Ore set****. If* S *satisfies O1 and O2, it is called a* ***right reversible*** *or* ***right denominator set****. There are analogous definitions for left everything.*

Remark 19.9: O1 allows us to pull denominators of fractions to the right: if $aS \cap sR \neq \varnothing$, then $at = sb$ for $t \in S, b \in R$. So using formal inverses, $s^{-1}a = bt^{-1}$.

Using O1 and O2, then R_S will consist of pairs $(a, s) \in R \times S$ modulo the equivalence that $(a, s) \sim (a', s')$ if there exist $u, u' \in R$ such that

$$au = a'u', \; su = s'u' \in S.$$

That is,

$$as^{-1} = (au)(su)^{-1} = (a'u')(s'u')^{-1} = a'(s')^{-1}.$$

This has a ring structure where $a \mapsto (a, 1)$ is a ring homomorphism.

Remark 19.10: Localization of a ring or a module can also be presented as a filtered colimit. We can create a diagram category D where the objects are S and $\mathrm{Hom}(s, t) = \{u \mid su = t\}$ and composition is given by $v \circ u = uv$. Then if O1 and O2 both hold, then D is filtered. Moreover, R_S is the filtered colimit $\lim_D R$. This shows that localization is exact because filtered colimits are (for abelian groups); also, it comes with the forgetful functor. We will prove this next lecture.