Lecture 20: Ore Localization, Goldie Theorem

Proposition 20.1: *Let S be a right reversible multiplicative subset in a ring R, i.e. it satisfies O1 and O2. Say that $(a, s) \sim (a', s')$ if there exist $t, t' \in R$ such that $at = a't'$ and $st = s't' \in S$ (that is, $a/s = a'/s'$). This is an equivalence relation on $R \times S$ and the map $(a, s) \mapsto as^{-1}$ is a bijection between $(R \times S)/\sim$ and the localization R_S.*

Proof. The relation is clearly reflexive and symmetric, we need to show transitivity. Suppose $(a, s) \sim (a', s')$, so $at = a't'$ and $st = s't' \in S$ for some $t, t' \in R$, and also $(a', s') \sim (a'', s'')$, so there exist $u, u' \in R$ such that $a''u = a'u', s''u = s'u' \in S$. We need to find $v, v'' \in S$ such that $av = a''v'', sv = s''v'' \in S$.
Apply O1 to $\alpha := s't', \sigma := s'u'$ to see that there exists $z_0 \in S, x_0 \in R$ such that $s't'z_0 = s'u'x_0$. Applying O2 to $s'(t'z_0 - u'x_0) = 0$, there exists some $r \in S$ such that $(t'z_0 - u'x_0)r = 0$. In other words, there exist elements $z \in S$, $x \in R$ satisfying $t'z = u'x$.
Therefore,

$$atz = a't'z = a'u'x = a''ux$$

with

$$stz - s''ux = s'(t'z - u'x) = 0 \Rightarrow s''ux = stz \in S.$$

Hence, $\sim$ is an equivalence relation.
To define a ring structure on the set of equivalence classes, write as^{-1} for the equivalence class of (a, s). To multiply $as^{-1} \cdot bt^{-1}$, find $c \in R, u \in S$ with $bu = sc$ and set

$$as^{-1} \cdot bt^{-1} = ac(tu)^{-1}.$$

To add $as^{-1} + bt^{-1}$, find s', t' such that $ss' = tt' \in S$ (these exist using O1), then

$$as^{-1} + bt^{-1} = (as')(ss')^{-1} + (bt')(tt')^{-1} = (as' + bt')(ss')^{-1}.$$

One can check that these are well-defined and produce an associative ring. Denote this ring by RS^{-1}. There is a map $R_S \to RS^{-1}$ since the map $R \to RS^{-1}$ sending $r \mapsto (r, 1)$ sends S to units. In the other direction, there is a map $RS^{-1} \to R_S$ sending $(a, s) \mapsto as^{-1}$. It's easy to see this map is a homomorphism and the two homomorphisms above are inverse isomorphisms. □

Corollary 20.2: *For a right denominator set $S \subset R$, the kernel of the canonical homomorphism $R \to R_S$ is the set of elements whose right annihilator intersects S.*

Proof. The kernel is the set of elements $a \in R$ such that $(a, 1) \sim (0, 1)$, which is true iff $as = 0$ for some $s \in S$. □

Definition 20.3: *An element of R is* ***regular*** *if it is neither a left nor right zero divisor.*

Corollary 20.4: *If S consists of regular elements, the natural map $R \to R_S$ is injective.*

20.1 Ore localization as a filtered colimit

Extending the remark 19.10 from last time, the localization can also be interpreted as a filtered colimit.

Recall from Definition 9.11 that a category D is filtered if $\mathrm{Ob}(D) \neq \varnothing$ and

- for every $a, b \in D$, there exists $c \in D$ such that $\mathrm{Hom}(a, c)$ and $\mathrm{Hom}(b, c)$ are nonempty
- for every pair of parallel morphisms $e, f : a \to b$, there exists $g : b \to c$ such that $g \circ e = g \circ f$.

Taking the filtered limit of abelian groups is exact and commutes with the filtered colimit of sets under the forgetful functor. The filtered colimit of sets can be described as follows: for a functor $F : D \to \mathrm{Set}$, its colimit is the quotient

$$\bigsqcup_{a\in \mathrm{Ob}(D)} F(a)/\sim$$

where $x \sim y$ for $x \in F(a), y \in F(b)$ if $y = F(e)(x)$ for some $e \in \mathrm{Hom}(a,b)$. (That is, there's an arrow in the image of F from x to y.)

As in the last lecture, we can create a diagram category D where the objects are S and $\mathrm{Hom}(s,t) = \{u \mid su = t\}$ and composition is given by $v \circ u = uv$.

Proposition 20.5: *If S is a right denominator set (i.e. both O1 and O2 hold), then D is filtered.*

Proof. First, D is nonempty because $1 \in S$.
For every $s, t \in \mathrm{Ob}(D) = S$, there exists a, b such that $sa = tb$ via O1, so $\mathrm{Hom}(s, sa)$ and $\mathrm{Hom}(t, tb)$ are nonempty. Two parallel morphisms $s \longrightarrow t$ are $a, b \in R$ such that $t = sa = sb$. Then by O2, $s(a-b) = 0$ implies there exists $u \in S$ such that $(a-b)u = 0$. So by composing the two parallel morphisms a, b with the morphism $t \longrightarrow tu$ given by u, we get the same morphism. □

Now for M a right R-module, define a functor $F_M : D \longrightarrow R^{\mathrm{op}}$-mod by sending every object to M and every morphism corresponding to $u \in R$ to right multiplication by u. Hence, R_S is the colimit of F_R. Therefore,

$$\operatorname{colim} F_M := M \otimes_R R_S =: M_S$$

is the localization of M at S, and $M \mapsto M_S$ is exact.

Remark 20.6: Ore conditions can also be generalized to categories: many important constructions involve inverting a class of morphisms in a category, and the generalization of the Ore conditions guarantees a manageable result. The construction of a **derived category** as a localization of the homotopy category of complexes is an example.

20.2 Ore domains

Definition 20.7: *A ring R is an* ***Ore domain*** *if it's a domain and $R\backslash\{0\}$ satisfies O1. In this case, R_S for $S = R\backslash\{0\}$ is clearly a skew field and $R_S = \mathrm{Frac}(R)$.*

Example 20.8: A free ring (e.g. over a field) with at least two generators is *not* an Ore domain: if x, y are free generators then $xR \cap yR = 0$.

Proposition 20.9: *Assume R is a domain.*

a) *(Goldie) Either R is a right Ore domain or it contains a free right ideal of infinite rank.*
b) *(Jategoankar) Say R is an algebra over a field k. Then either R is a left and right Ore domain or it contains a free ring $k\langle x, y\rangle$.*

Proof. a) Suppose R is not a right Ore domain, so there exist a, b such that $aS \cap bR = \varnothing$ (recall that $S = S \setminus \{0\}$). Then we claim that $a, ba, b^2a, \ldots,$ is right independent over R. Otherwise, we could find $\{r_i\}$ such that

$$\sum_{i=0}^{n} b^i a r_i = 0 \Rightarrow -ar_0 = b\left(\sum_{i=1}^{n} b^{i-1} a r_i\right),$$

contradiction (note that we can assume that $r_0 \neq 0$ i.e. $-r_0 \in S$).

b) Suppose R is not a right Ore domain and pick x, y such that $xR \cap yR = 0$. Let $f(x,y) = a + xf_1 + yf_2$ be a minimal relation where $a \in k$. If $a = 0$, then $xf_1 = yf_2 \neq 0$ but $xR \cap yR = 0$, contradiction. If $a \neq 0$, multiplying everything by y on the right, we have $ay + xf_1y + yf_2y = 0$. Since $a \in k$, $ay = ya$ and $x(f_1y) = y(a + f_2y)$. These are again both nonzero: if $f_1y = 0$, then $f_1 = 0$ because R is a domain, so $yf_2 + a = 0$, so y is invertible. Then $yR = R$, so $xR \cap yR \neq 0$, contradiction. Likewise, $a + f_2y \neq 0$. So $xR \cap yR$ has a nonzero element, a contradiction. Thus x, y generate a free algebra.

The same argument works if R is not a left Ore domain. □

20.3 Growth of algebras

Let A be a finitely generated k-algebra for a field k. Let V be a (finite-dimensional) vector space of generators for A, so we have an onto map $TV \twoheadrightarrow A$ where TV is a tensor algebra. Let $A^V_{\leqslant n}$ be the image of $\bigoplus_{i \leqslant n} V^{\otimes i}$ and set

$$d_V(n) := \dim_k(A^V_{\leqslant n}).$$

For a different space of generators W, $d_W \neq d_V$, but $d_W(n) \leqslant d_V(n_0 n)$ always for some fixed n_0 because $A^W_{\leqslant n} \subset A^V_{\leqslant n_0 n}$ for some n_0.

So say that two (monotone) functions f, g on $\mathbb{N}$ are equivalent if there exists n_0 such that

$$f(n) \leqslant g(n_0 m),\ g(n) \leqslant f(n_0 m).$$

So the equivalence class of $d_V(n)$ is independent of the choice of V.

Definition 20.10: *We say that A has **exponential growth** if $d(n) \geqslant c\alpha^n$ for some constants $\alpha > 1, c$. If A does not have exponential growth, it necessarily has **subexponential growth**, i.e. for all $\alpha > 1$, $f(n)/\alpha^n \to 0$.*

Example 20.11: If A contains a free algebra, then A has exponential growth.

Corollary 20.12 (of Proposition 20.9): *If A is a domain of subexponential growth, then A is an Ore domain.*

Example 20.13: The Weyl algebra

$$W_n = k\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle / ([x_i, y_j] = \delta_{ij}, [x_i, x_j] = [y_i, y_j] = 0)$$

and $U(\mathfrak{g})$ for $\mathfrak{g}$ a finite-dimensional Lie algebra are domains of polynomial, hence subexponential, growth, and therefore are Ore domains.

20.4 Semi-prime rings and Goldie's theorem

Recall that an element is regular if it's neither a left or right zero divisor.

Remark 20.14: For a regular element, left invertibility is equivalent to right invertibility, since $sr = 1 \Rightarrow rsr = r \Rightarrow rs = 1$.

Definition 20.15: *A ring is called **prime** if $IJ \neq 0$ for any two nonzero two-sided ideals $I, J \subset R$. It is **semi-prime** if $I^2 \neq 0$ for any nonzero two-sided ideal $I \subset R$.*

Recall that a ring is semi-primitive if its Jacobson radical vanishes, which is equivalent to the existence of a faithful semisimple (either left or right) module.

Proposition 20.16: *Every semi-primitive ring is semi-prime.*

Proof. Suppose $I \subset R$ is a nonzero two-sided ideal, and R is semi-primitive. So we can find an irreducible R-module L such that $IL \neq 0$. Then from the density theorem, it follows that we can find $x \in I, v \in L, v \neq 0$ where $xv = v$. Hence, $x^2 \neq 0$. □

The converse is not true, but we do have the following:

Theorem 20.17 (Goldie): *If R is a semi-prime right Noetherian ring, then the set S of all regular elements satisfies (right) O1, and $Q = R_S$ is an Artinian semisimple ring.*

Corollary 20.18: *If R is left or right Noetherian, it admits a homomorphism to* $\mathrm{Mat}_n(D)$*, so it satisfies the IBN.*

Proof. If R is right Noetherian, then $\bar{R} := R/J(R)$ is semi-primitive and right Noetherian, hence semi-prime. By Goldie's theorem, $\bar{R}_S$ is Artinian semisimple, so $\bar{R}_S = \prod_{i=1}^n \mathrm{Mat}_{d_i}(D_i)$. Hence

$$R \to \bar{R} \to \bar{R}_S \to \mathrm{Mat}_{d_1}(D_1)$$

is the desired homomorphism. □

The idea of the proof of the theorem is that sR is "too big" to miss aS; we need a notion of size.

Definition 20.19: *Let M be a right R-module. A submodule $E \subset M$ is* ***essential*** *if for all nonzero $N \subset M$, $N \cap E \neq 0$. That is, every nonzero submodule in M has a nonzero intersection with N. We say that M is* ***uniform*** *if $M \neq 0$ and every nonzero submodule in M is essential.*

Example 20.20: If M is of finite length, $E \subset M$ is essential iff $E \supset \mathrm{Soc}(M)$ and M is uniform iff $\mathrm{Soc}(M)$ is simple. For example, for $R = k[t]$, $M = k[t]/(t^n)$ is uniform. Another example is a domain R considered as a (right) module over itself.

Lemma 20.21: *If $N \subset M$ is a submodule, then there exists a submodule $N' \subset M$ such that $N \oplus N'$ is an essential submodule in M. N' is called the* ***essential complement*** *of N.*

Proof. Consider all submodules with zero intersection with N. Then the condition of Zorn's Lemma holds, so there exists a maximal element N' in this set. Then $N \oplus N'$ is essential in M. □

The measure of size we will use is the maximal number of uniform submodules of M such that their direct sum is also a submodule of M.

Proposition 20.22:

a) *Let M be a Noetherian module. Then it contains an essential submodule that is a sum of uniform submodules, $E = \bigoplus_{i=1}^n U_i$, E essential and U_i uniform.*
b) *The number of uniform summands is independent of choices and is the* ***Goldie rank*** *or* ***uniform dimension****.*
c) *Every submodule of full Goldie rank is essential. That is, if $M \supset N$ and $\mathrm{Grank}(M) = \mathrm{Grank}(N)$, then N is essential in M.*

Corollary 20.23: *If $s \in R$ is a regular element, then $sR \subset R$ is an essential ideal.*

Lemma 20.24: *The preimage of an essential submodule is essential.*

Proposition 20.25: *An essential right ideal in a semi-prime right Noetherian ring contains a regular element.*

Next time, we will prove these and discuss other facts about essential modules.

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Spring 2023