Lecture 21: Goldie Rank and Goldie Theorem

21.1 More on essential modules

Corollary 21.1: *A module M has no proper essential submodules iff it is semisimple.*

Proof. We proved that a module M is semisimple iff every submodule N has a direct complement. So if $N \subset M$, we know it has an essential complement N' such that $N \oplus N'$ is essential. If M has no proper essential submodules, then $N \oplus N' = M$ and M is semisimple. If M is semisimple, every submodule's direct complement doesn't intersect it, so there are no proper essential submodules. □

Lemma 21.2:
a) *If $M \supset N \supset P$ with N essential in M and P essential in N, then P is essential in M.*
b) *The preimage of an essential submodule is essential.*
c) *If $N_1 \subset M_1, N_2 \subset M_2$ are essential, then $N_1 \oplus N_2 \subset M_1 \oplus M_2$ is essential.*

Proof. *a)* If $S \subset M$ has nonzero intersection with N (use that $N \subset M$ is essential), then $S \cap N \subset N$ has nonzero intersection with P (use that $P \subset N$ is essential).
b) Let $\varphi\colon M \to N$ and $E \subset N$ essential. Suppose $V \subset M$ is a nonzero submodule. Then either $V \subset \ker(\varphi)$ or $\varphi(V)$ is nonzero. If $V \subset \ker(\varphi)$, $V \subset \varphi^{-1}(E)$. If $\varphi(V) \neq 0$, then $\varphi(V) \cap E \neq 0$, so $V \cap \varphi^{-1}(E) \neq 0$.
c) By a), it's enough to consider $M_1 = N_1$. Then $M_1 \oplus N_2$ is the preimage of N_2 under the projection $M_1 \oplus M_2 \twoheadrightarrow M_2$, so it is essential by b).

□

21.2 Goldie rank

Definition 21.3: *A module M has* ***finite Goldie rank*** *if it does not contain an infinite sum of nonzero submodules.*

Example 21.4: If M is Noetherian, it has a finite Goldie rank. In fact, one can restate the finite Goldie rank condition as the condition that split increasing chains of submodules should stabilize, where a chain of submodules M_i splits if M_i has a direct complement in M_{i+1} for all i.

Proposition 21.5: *A finite Goldie rank module contains an essential submodule which is a finite sum of uniform submodules.*

Proof. Suppose for contradiction that M does not contain such an essential submodule. Then M is not uniform, so it has a nonessential submodule N_1 with essential complement C_0. Then if both N_1, C_0 contain essential submodules E_1, E_2 respectively, then $E_1 \oplus E_2$ is essential in M. So WLOG suppose C_0 does not contain an essential submodule. Then repeat the same argument for C_0; we get two submodules N_2, C_1 where $N_2 \oplus C_1 \subset C_0$ and C_1 does not contain an essential submodule. Thus by induction we get $C_1, C_2, \ldots$ where $C_i \supset N_i \oplus C_{i+1}$. Hence $N \supset N_1 \oplus N_2 \oplus \cdots$, contradicting the assumption. □

Theorem 21.6: *Suppose M has finite Goldie rank and contains $E = \bigoplus_{i=1}^m U_i$ an essential sum of uniform submodules. If $M \supset N = \bigoplus_{i=1}^n N_i$ with $N_i \neq 0$, then $n \leqslant m$. If $m = n$, then N is essential and each N_i is uniform.*

Proof. First, $N' := \bigoplus_{i=2}^n N_i$ is not essential. then we claim that $N' \cap U_i = 0$ for some i. Otherwise, $N' \cap U_i \neq 0$ is essential in U_i, so by the lemma $\bigoplus_{i=1}^m (N' \cap U_i)$ is essential in M and N' is essential in M.
WLOG say that $N' \cap U_1 = 0$. Then $U_1 \oplus N_2 \oplus \cdots \oplus N_n \subset M$. Continuing inductively, with possible reindexing, $U_1 \oplus \cdots \oplus U_i \oplus N_{n-i} \oplus \cdots \oplus N_n \subset M$. Therefore, $n \leqslant m$.
If $m = n$, then N is essential. If not, we'd have an essential complement N' and $N \oplus N'$ would be a sum of $n+1$ nonzero submodules, contradiction. Likewise, each N_i is uniform: otherwise, it would have a nonessential submodule N_i' with essential complement N_i'', so we would again get a direct sum of $n+1$ submodules. □

Corollary 21.7: *If M has finite Goldie rank n, then every submodule in M with the same Goldie rank n is essential.*

Corollary 21.8: *The Goldie rank can also be defined as the maximal number of $M_i \neq 0 \subset M$ such that $\bigoplus_i M_i \subset M$.*

Example 21.9: For semisimple modules, the Goldie rank is the number of simple summands.

21.3 Regular elements in essential ideals

Remark 21.10: Suppose $S \subset R$ consists only of regular elements. Then the localization of an essential (resp. uniform) ideal at S is essential (resp. uniform).

Theorem 21.11: *An essential right ideal in a semi-prime, right Noetherian ring contains a regular element.*

This will imply the first statement in Goldie's theorem: let S be the regular elements. Given $s \in S$, $sR \cong R$ so it has the same Goldie rank as R (as a right module over itself) and is essential in R (use Corollary 21.7). Hence, for any $a \in R$, the preimage of sR under the map $x \mapsto ax$ is an essential right ideal (Lemma 21.2) and contains a regular element t. Thus $aS \cap sR \neq \emptyset$, which implies O1; O2 is vacuous for regular elements.

To prove the theorem, we first start with a weaker claim.

Lemma 21.12: *Let R be a right Noetherian, semi-prime ring and $I \subset R$ an essential right ideal. Then the left annihilator of I is zero.*

Proof. Let J be the left annihilator of I. We know $J^2 \neq 0$ because R is semi-prime (if $I^2 = 0$, then $(JR)^2 = 0$ for the two-sided ideal JR). Replace I by $\mathrm{rAnn}(J)$; WLOG we can assume that I is maximal among right annihilators using the Noetherian property.

Since $J^2 \neq 0$, pick $x, y \in J$ such that $xy \neq 0$. Then $yR \cap I \neq 0$ since I is essential, so there exists r with $yr = z \in I$ and $xyr = 0$. Then

$$r \notin \mathrm{rAnn}(I), r \in \mathrm{rAnn}(xy) \Rightarrow \mathrm{rAnn}(xy) \supsetneq \mathrm{rAnn}(y) \supset I$$

which contradicts the maximality of I. □

Proposition 21.13: *Any right ideal I contains an element x with* $\mathrm{rAnn}(x) \cap I = 0$.

This proposition implies Theorem 21.11. Let I be an essential ideal. Then we can find $r \in I$ with $\mathrm{rAnn}(x) \cap I = 0$. Since I is essential, this means $\mathrm{rAnn}(r) = 0$ and rR is free. In particular, it has the same Goldie rank as R, so rR is essential in R. Then by the lemma, $\mathrm{lAnn}(rR) = \mathrm{lAnn}(r) = 0$. So r is regular.