Lecture 22: Goldie Theorem, PI Rings

22.1 Finishing up Goldie Theorem

Proof (of Proposition 21.13). First, we prove the claim when I is uniform (see Definition 20.19). Again, $I^2 \neq 0$ since R is semi-prime, so pick $x, y \in I$, $xy \neq 0$. Then we claim that $\mathrm{rAnn}(x) \cap I = 0$. Otherwise, $\mathrm{rAnn}(x) \cap I$ is essential in I. Consider the homomorphism of (right) R-modules $L_y : R \to I$ given by $z \mapsto yz$. It follows from Lemma 21.2 that the preimage $L_y^{-1}(\mathrm{rAnn}(x) \cap I)$ is essential in R. So $\{z \in R \mid yz \in \mathrm{rAnn}(x)\}$ is essential in R. But then its left annihilator is zero by the above lemma, but $x \neq 0$ is in the annihilator, contradiction.
In general, choose a maximal subideal $J \subset I$ such that there exists $v \in J$ with $\mathrm{rAnn}(v) \cap J = 0$ (via the right Noetherian property). If $\mathrm{rAnn}(v) \cap I \neq 0$, pick a uniform ideal $U \subset \mathrm{rAnn}(v) \cap I$. There exists $u \in U$ with $\mathrm{rAnn}(u) \cap U = 0$. Set $x = u + v$.
Since $U \subset \mathrm{rAnn}(v)$, $U \cap J = 0$. So if $x \in \mathrm{rAnn}(u + v)$, then $x \in \mathrm{rAnn}(u) \cap \mathrm{rAnn}(v)$. Suppose $x = u' + v' \in U \oplus J$. Then $uu' + uv' = 0, vu' + vv' = 0$. But $vu' = 0$ since $U \subset \mathrm{rAnn}(v)$, so $vv' = 0 \Rightarrow v' = 0$. So $uu' = 0$ and $u' = 0$ by assumption on u. Thus, $J \oplus U$ is a larger subideal in I containing an element $u + v$ whose right annihilator has zero intersection with the ideal, contradicting the maximality of J. ◻

Proof (of Theorem 20.17). To finish proving the Goldie theorem, we need to show that R_S is Artinian semisimple. This is equivalent to R_S being semisimple as a right module over itself, which is equivalent to saying that R_S has no proper essential ideals. Suppose that $I \subset R_S$ is essential. Then $I \cap R$ is essential in R: $R \hookrightarrow R_S$ because S consists of regular elements, so the preimage of $I \subset R_S$, which is $I \cap R$ is essential.
Then $I \cap R$ contains a regular element (Theorem 21.11), i.e., $R \cap I \cap S$ is nonempty, so $I = R_S$. ◻

22.2 Goldie rings

The statement of Goldie's theorem required R to be semi-prime right Noetherian. However, the proof only uses the fact that R has 1) finite Goldie rank as a right module over itself (split ascending chains of right ideals stabilize) and 2) chains of right annihilators stabilize.

This is because even though we invoked the Noetherian property to find a maximal ideal $J \subset I$ with $v \in J$ such that $\mathrm{rAnn}(v) \cap J = 0$, the proof found an ideal of the form $J \oplus U$, so it suffices to use that split chains terminate.

Definition 22.1: *If R has finite Goldie rank as a right module over itself and chains of right annihilators stabilize, we say that R is a* ***(right) Goldie ring****.*

Example 22.2: Not every right Goldie ring is right Noetherian. For example, every commutative domain where every annihilator of a nonzero element is zero and every nonzero ideal is essential is a right Goldie ring but not necessarily right Noetherian.

22.3 Applications of Goldie's Theorem

Proposition 22.3: *Let R be a semi-prime Goldie ring and S the set of its regular elements. Then if $I \subset J$ is an essential subideal, the localizations I_S and J_S coincide. Also, if I is uniform then I_S is irreducible.*

Proof (Sketch). Essential embeddings and uniformity survive after localization. Over semi-simple Artinian rings, uniform modules are irreducible and essential embeddings are isomorphisms. □

Hence, Goldie rank is a measure of the size of an infinite-dimensional algebra (say, for algebras over a field) and it's an interesting question to understand it better and compare it with other measures.

Example 22.4: What is the Goldie rank of R as a module over itself? For example, if R is prime (in particular, if it is primitive), then $R_S \cong \mathrm{Mat}_n(D)$, and the Goldie rank will be n.

A very interesting story is related to the study of this invariant for $R = U(\mathfrak{g})/I$ where $\mathfrak{g}$ is a complex simple finite-dimensional Lie algebra (e.g. $\mathfrak{sl}(n)$) and I is a primitive ideal. Then the answer is given by the "Goldie rank polynomial"; the classification of ideals involves a parameter λ on which the answer depends polynomially. This is largely understood due to the work of various authors, including David Vogan, George Lusztig, Tony Joseph, and, more recently, Ivan Losev.

Another famous question related to noncommutative localization and Lie theory is the Gelfand-Kirillov conjecture. This states that for a large class of Lie algebras, including those mentioned above, the fraction field of $U(\mathfrak{g})$ (a domain of polynomial growth, hence an Ore domain) is isomorphic to the fraction field of a ring of the form $W_n[x_1, \dots, x_r]$ where W_n is the Weyl algebra. This turned out to be false in general, but true for $\mathfrak{g} = \mathfrak{sl}(n)$. However, if $\bar{U} = U(\mathfrak{g})/\mathfrak{m}U(\mathfrak{g})$, where $\mathfrak{m}$ is a maximal ideal in the center of $U(\mathfrak{g})$, then the fraction field of $\bar{U}$ is indeed isomorphic to the fraction field of W_n for every simple complex Lie algebra.

22.4 PI rings

Definition 22.5: *A ring R is a* ***polynomial identity (PI) ring*** *if there exists a nonzero element in the free algebra $P \in \mathbb{Z}\langle x_1, \dots, x_n\rangle$ such that $P(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in R$ (i.e., there is a polynomial identity that all elements satisfy).*
Likewise, if A is an algebra over a field (or commutative ring) k, it is a ***polynomial identity (PI) algebra*** *if there exists a nonzero $P \in k\langle x_1, \dots, x_n\rangle$ such that any evaluation of P in A vanishes.*

Example 22.6: Commutative rings are PI rings: take $P(x, y) = xy - yx$.

Example 22.7: Boolean rings (rings where every element is idempotent) are also PI rings with $P(x) = x^2 - x$.

Example 22.8: Let
$$S_n(x_1,\dots,x_n) = \sum_{\sigma\in S_n} (-1)^{|\sigma|} x_{\sigma(1)}\cdots x_{\sigma(n)}.$$
We claim that this holds in every finite-dimensional algebra A over a field k of char $k \neq 2$ when $n > \dim_k(A)$. This is because evaluation of S_n is a skew-symmetric multilinear functional, hence is a map $\Lambda^n(A) \to A$. But if $n > \dim_k(A)$, then $\Lambda^n(A) = 0$.

22.5 Amitsur-Levitzki Theorem

Theorem 22.9 (Amitsur-Levitzki): *The identity S_{2n} holds in the ring $\mathrm{Mat}_n(R)$ for any commutative ring R. Moreover, no (nonzero) homogeneous identity of smaller degree holds (assuming $R \neq 0$).*

The second part of the theorem is easier and follows from the next two lemmas.

Lemma 22.10 (Staircase Lemma): $\mathrm{Mat}_n(R)$ *does not satisfy a multilinear identity of degree $d < 2n$.*

Proof. Consider the following $2n-1$ elementary matrices:
$$e_{11}, e_{12}, e_{21}, e_{22}, \dots, e_{n-1,n-1}, e_{n-1,n}, e_{n,n}$$
Their product in this order is an elementary matrix, namely e_{1n}, but their product in any other order vanishes. The first r matrices in that list for $r < 2n-1$ satisfy the same property.
A multilinear polynomial is a linear combination of multi-homogeneous monomials with coefficients in R. If a degree r monomial $x_1\cdots x_r$ is in the polynomial, substitute the above elementary matrices for x_i and zero for the other variables (if any). Then our sum has exactly one nonzero summand, so the sum is nonzero. □

Lemma 22.11:
a) *If a ring satisfies an identity P of degree d, then it satisfies a multilinear identity of the same degree.*
b) *If an algebra A over an infinite field k satisfies a polynomial identity $P = \sum P_d$ where P_d is homogeneous of degree d, then each P_d is also an identity satisfied by A.*

Proof. *a)* Let $P = P(x_1,\dots,x_n)$ be a degree d identity. We do double induction on the top degree of P in each variable and the number of variables in which it has that degree. Suppose $r > 1$ is the top degree and WLOG that P has degree r in x_1. Then consider
$$Q(x_0,\dots,x_n) = P(x_0+x_1,x_2,\dots,x_n) - P(x_0,x_2,\dots,x_n) - P(x_1,\dots,x_n).$$
Q holds in our ring and has degree less than r in both x_0, x_1. For the other variables, their degree is most that of P. Note that Q is not identically zero: this is because for monomials M of degree d, the noncommutative polynomials
$$M' = M(x_0+x_1,x_2,\dots,x_n) - M(x_0,x_2,\dots,x_n) - M(x_1,\dots,x_n)$$
are linearly independent over R. This is because the monomials in M' which are linear in x_0 will enter M' with multiplicity 1, and we can reconstruct M from such a monomial by replacing x_0 by x_1.
Therefore, by induction we can find an identity P which has degree one in each variable. Suppose there is a variable x_i appearing in P in which P is not linear (so x_i appears in some monomials but not in others). Then
$$P(x_1,\dots,x_i,\dots,x_n) - P(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n)$$
is also an identity and is nonzero and linear in x_i. Repeating this inductively, we get a multilinear identity of the same total degree.
b) For $\lambda \in k$, $P_\lambda = \sum \lambda^d P^d$ is also a polynomial identity. Choosing distinct $\lambda_1,\dots,\lambda_n$ with $n > \deg(P)$, the linear span of P_{λ_i} will contain P_d because the Vandermonde determinant doesn't vanish.
□

This furnishes a proof of the second part of the theorem.

22.6 Amitsur-Levitzki Theorem and the cohomology of $\mathfrak{gl}(n)$

We will sketch the proof of the Amitsur-Levitzki Theorem via this Lie algebra cohomology story. To simplify notation, we work over $\mathbb{C}$.

Notice that the identity S_{2n} holding in $\mathrm{Mat}_n(k)$ is equivalent to

$$\mathrm{Tr}(S_{2n+1}(x_1, \dots, x_{2n+1})) = 0$$

for all $x_1, \dots, x_{2n+1} \in \mathrm{Mat}_n(k)$. To see this, note that trace is cyclically invariant ($\mathrm{tr}(abc) = \mathrm{tr}(cab)$, etc.), so for each monomial in S_{2n+1}, we can cyclically permute the variables until x_1 is at the left. Factoring x_1 out, we obtain $\mathrm{Tr}(x_1 S_{2n}(x_2, \dots, x_{2n+1})) = 0$. Since the trace pairing is nondegenerate, this implies that $S_{2n}(x_2, \dots, x_{2n+1}) = 0$.

Now view $\mathrm{Mat}_n(\mathbb{C})$ as a Lie algebra, so $\mathfrak{gl}_n(\mathbb{C})$. The multilinear functional

$$(x_1, \dots, x_{2i-1}) \mapsto \mathrm{Tr}(S_{2i-1}(x_1, \dots, x_{2i-1}))$$

defines an element

$$\sigma_i = \sigma_{i,n} \in \Lambda^{2i-1}\mathfrak{g}^*$$

invariant under conjugation by $G = \mathrm{GL}_n(\mathbb{C})$, so $\sigma_i \in (\Lambda^{2i-1}\mathfrak{g}^*)^G$. For $G = \mathrm{GL}_n(\mathbb{C})$ and other complex reductive groups, there are isomorphisms

$$(\Lambda^\bullet \mathfrak{g}^*)^G \cong H^\bullet(\mathfrak{g}) \cong H^\bullet(G, \mathbb{C}) \cong H^\bullet(K, \mathbb{C}).$$

Here $H^\bullet(\mathfrak{g})$ is the **Lie algebra cohomology**, i.e. $\mathrm{Ext}^\bullet_{U(\mathfrak{g})}(\mathbb{C}, \mathbb{C})$ (parallel to the definition of group cohomology). $H^\bullet(G, \mathbb{C})$ is the cohomology of G viewed as a topological space, while $K \subset G$ is a maximal compact subgroup and $H^\bullet(K, \mathbb{C})$ is the cohomology for K viewed as a topological space. For $G = \mathrm{GL}_n(\mathbb{C})$, the maximal compact subgroup K is the group $U(n)$ of unitary matrices, and

$$H^*(U(n), k) = \Lambda[c_{1,n}, c_{2,n}, \dots, c_{n,n}], \ \deg(c_{i,n}) = 2i - 1.$$

This is graded and skew-commutative so $c_i^2 = 0$. This follows from induction and the fact that $U(n)/U(n-1) = S^{2n-1}$ (the $(2n-1)$-dimensional sphere). The restriction map

$$H^\bullet(\mathfrak{gl}(n)) \to H^\bullet(\mathfrak{gl}(n-1))$$

sends $c_{i,n} \mapsto c_{i,n-1}$ when $i \leqslant n-1$ and $c_{n,n} \mapsto 0$.

This gives a proof of the Amitsur-Levitski Theorem as follows:

We want to show that $\sigma_{i,n} = 0$ for $i > n$. We induct on n, so assume $\sigma_{i,n-1} = 0$ for $i > n-1$. So in particular

$$\sigma_{n+1,n} \in \ker(H^{2n+1}(\mathfrak{gl}(n)) \to H^{2n+1}(\mathfrak{gl}(n-1))).$$

We claim this map is injective: the kernel of the restriction map $H^\bullet(\mathfrak{gl}(n)) \to H^\bullet(\mathfrak{gl}(n-1))$ is generated by an element of degree $2n-1$ and $H^2(\mathfrak{gl}(n)) = 0$, so there is nothing in the kernel in degree $2n+1$. So $\sigma_{n+1,n} = 0$.

It remains to show that $\sigma_{i,n} = 0$ for $i > n+1$. The vanishing of $\sigma_{i,n}$ is equivalent to S_{2i} being an identity in $\mathrm{Mat}_n(\mathbb{C})$. But if the identity S_m holds, so does S_p for $p > m$ because one can sum over the symmetric group Σ_p by first summing over the Σ_m-cosets in Σ_p. This completes the induction.

Remark 22.12: $H^\bullet(\mathfrak{gl}(n))$ is in fact freely generated by $\sigma_{1,n}, \dots, \sigma_{n,n,}$.