Lecture 23: Another Proof of Amitsur-Levitski, PI Algebras

23.1 Proof of Amitsur-Levitski Theorem

We now give a self-contained proof of the first part of Theorem 22.9 using Cayley-Hamilton, due to Rossett.

Theorem 23.1 (Cayley-Hamilton): *Let $x \in \mathrm{Mat}_n(R)$ be an $n \times n$ matrix with coefficients in R and $P_x(t) \in R[t]$ be its characteristic polynomial. Then $P_x(x) = 0$.*

Proof (Sketch). The "easiest to remember" proof is to first reduce to $R = \mathbb{Z}$ by noting that the entries $P_x(x) = 0$ will be polynomials in the entries of x with integer coefficients. Then it suffices to show this for $R = \mathbb{C}$. But over $\mathbb{C}$ all matrices can be put in Jordan normal form, and for such a matrix $P_x(x) = 0$. □

Proof. A more aesthetically appealing proof: every matrix A has an adjoint matrix B such that $AB = BA = \det(A) \cdot I_n$ where I_n is the identity $n \times n$ matrix. Then in $\mathrm{Mat}_n(R[t])$, letting $A = t \cdot I_n - x$, by definition $\det(A) = P_x(t)$ and there exists B such that

$$BA = AB = P_x(t) \cdot I_n.$$

Let $\mathrm{Mat}_n(R[t]) = \mathrm{Mat}_n(R)[t]$ act on $\mathrm{Mat}_n(R)$ where $\mathrm{Mat}_n(R)$ acts via left multiplication and t acts by right multiplication by x. Then

$$A.I_n = 0 \Longrightarrow (P_x(t) \cdot I_n).I_n = 0$$

so $P_x(x) = 0$. □

Corollary 23.2: *If $P_x(t) = t^n$, then $x^n = 0$.*

Proof (of Theorem 22.9). It suffices to consider $R = \mathbb{Z}$ since multilinear identities are inherited by the extension of scalars. Since $\mathrm{Mat}_n(\mathbb{Z}) \subset \mathrm{Mat}_n(\mathbb{Q})$, it is enough to consider $R = \mathbb{Q}$. We will show for a certain matrix x that $\mathrm{Tr}(x^i) = 0$ for $i = 1, \ldots, n-1$, which will imply that $P_x(t) = t^n$.
Consider an auxiliary ring $\Lambda = \Lambda^\bullet(\mathbb{Q}^{2n})$ (exterior algebra of $\mathbb{Q}^{2n}$) where $\mathbb{Q}^{2n}$ has basis $\varepsilon_1, \ldots, \varepsilon_{2n}$. For $x_1, \ldots, x_{2n} \in \mathrm{Mat}_n(\mathbb{Q})$ let

$$\boldsymbol{x} = \varepsilon_1 x_1 + \cdots + \varepsilon_{2n} x_{2n} \in \mathrm{Mat}_n(\Lambda) = \mathrm{Mat}_n(\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda.$$

Amitsur-Levitski will hold iff

$$\boldsymbol{x}^{2n} = \varepsilon_1 \wedge \cdots \wedge \varepsilon_{2n} \cdot S_n(x_1, \ldots, x_n) = 0.$$

Notice that the even-degree wedges Λ^{ev} form a commutative ring, so Cayley-Hamilton applies here! Decompose

$$\Lambda^{\mathrm{ev}} = \mathbb{Q} \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^{2n}.$$

So it remains to check that the coefficients of the characteristic polynomial of $\boldsymbol{x}^2$ vanish, i.e. $\mathrm{Tr}(\boldsymbol{x}^{2i}) = 0$. But this is true because

$$\mathrm{Tr}(x_1 \cdots x_{2i}) = \mathrm{Tr}(x_{2i} x_1 \cdots x_{2i-1})$$

and this cycle is an odd permutation because the number of letters is even. □

23.2 Primitive algebras and Kaplansky's theorem

Theorem 23.3 (Kaplansky): *Let A be a primitive PI algebra over a field k with a homogeneous identity of degree d. Then it is simple of degree $m \leqslant d/2$ over its center, which is a (possibly different) field K.*

Proof. Let L be a faithful simple module over A and $D := \mathrm{End}_A(L)$.
First, $\dim_D(L) \leqslant d/2$. If not, the image of $A \subset \mathrm{End}(L)$ would contain $\mathrm{Mat}_n(D)$ as a subquotient for $n > d/2$ (pick some collection of $n > d/2$ linearly independent vectors in L and consider the subalgebra A' of A that preserves vector space generated by these vectors, it then follows from the density theorem 3.3 that A' surjects onto $\mathrm{Mat}_n(D)$), which contradicts the easy part of Amitsur-Levitski.
Then we claim that D is finite over its center K. If not, pick a maximal commutative subfield $F \subset D$, which exists

by Zorn's lemma. As we discussed earlier, WLOG we can assume the identity is multilinear, so it's inherited by extension of scalars and also holds in $F \otimes_K D$. By Azumaya-Nakayama, $F \otimes_K D$ is a simple ring. Moreover, by having D act by left multiplication and F by right multiplication, we get an action of $F \otimes_K D$ on D, and D is a simple module over $F \otimes_K D$ with

$$\mathrm{End}_{F\otimes_K D}(D) = Z_D(F) = F.$$

So by the argument in the previous paragraph, D is finite-dimensional over F. $F \otimes_K D \subset \mathrm{End}_F(D)$ is finite over K because $F \otimes_K D$ simple impiles D is a faithful $F \otimes_K D$-module. Thus, D is finite over K.
Finally, to get the degree bound, let E be a splitting field of D. Then

$$\mathrm{Mat}_n(D) \otimes_K E = \mathrm{Mat}_{n-\deg D}(E) \Rightarrow 2n\deg(D) \leqslant d$$

via the easy part of Amitsur-Levitski. □

23.3 Prime PI algebras and Posner theorem

Theorem 23.4 (Posner): *Let A be a prime PI algebra. Then its center $Z = Z(A)$ is a domain. Moreover, $A \otimes_Z \mathrm{Frac}(Z) \cong \mathrm{Mat}_n(D)$ for some skew field D that is finite-dimensional over $K = \mathrm{Frac}(Z)$.*

The proof follows from another fact about semi-prime PI algebras, which follows from Kaplansky's Theorem.

Theorem 23.5 (Rowen): *Let A be a semi-prime PI algebra. Then every nonzero two-sided ideal meets the center.*

Corollary 23.6: *A prime PI ring A whose center is a field K is a central simple algebra over K.*

Proof. By Rowen's theorem, every nonzero two-sided ideal in A meets Z. Thus, A is simple, and Kaplansky's theorem shows A is finite-dimensional over K. □

Proof (of Theorem 23.4). Z is a domain, since if $z_1 z_2 = 0$ for central elements $z_1, z_2 \in Z$, then $Az_1 \cdot Az_2 = 0$, contradiction. Homogeneous polynomial identities are inherited by extension of scalars, so $A \otimes K$ is simple by the Corollary 23.6. □

23.4 Central polynomials

The proof of Rowen's theorem is via central polynomials, which are noncommutative polynomials that sometimes take values in the center. We will be interested in Razmyslov's central polynomials, which, when you plug in $n \times n$ matrices, give back a scalar matrix (and are not identically zero).

First, we start with a linear algebra construction. Recall that $M = \mathrm{Mat}_n(k)$ has a nondegenerate trace pairing

$$\langle x, y\rangle = \mathrm{Tr}(xy).$$

This corresponds to the element $\tau \in M \otimes M$, $\tau = \sum_i m_i \otimes m_i^*$ (coevaluation) where the m_i and m_i^* are dual bases.

Lemma 23.7: *For $x \in \mathrm{Mat}_n(k)$, $\sum_i m_i x m_i^* = \mathrm{Tr}(x) I_n$.*

Proof. We use the cyclicity of trace. Let $\mu(x)$ denote the LHS. Since $\langle ca, b\rangle = \langle a, bc\rangle$, we have

$$c\mu(x) = \mu(x)c.$$

Likewise, since $\langle ac, b\rangle = \langle a, cb\rangle$,

$$\mu(cx) = \mu(xc)$$

for all $c \in \mathrm{Mat}_n(k)$. Therefore, there exists $\lambda \in k$ such that

$$\mu(x) = \lambda\, \mathrm{Tr}(x) I_n.$$

If we plug in e_{11} for x, we see that the RHS is I_n, so $\lambda = 1$. □

Now we will look for noncommutative polynomials in n^2 variables such that $P(m_1, \ldots, m_{n^2})$ is an element of the dual basis.

Definition 23.8: *The **Capelli polynomial** is*

$$C(x_1, \ldots, x_N, y_0, \ldots, y_N) = \sum_{\sigma \in \Sigma_n} (-1)^{|\sigma|} y_0 x_{\sigma(1)} y_1 \cdots x_{\sigma(N)} y_N.$$

It's like S_N, but it inserts "separating" variables y_i.

Lemma 23.9: *Let a_i be a basis of $\mathrm{Mat}_n(k)$ and a_i^* be its dual basis (w.r.t. the trace pairing). Define $C_i := \tau_i(C)$ where τ_i is a linear endomorphism on the space of multilinear noncommutative polynomials defined by $\tau_i(ux_iv) = vu$. Then for $b_0, \ldots, b_{n^2}$ any matrices,*

$$\mathrm{Tr}(C(a_k, b_l))a_i^* = C_i(a_k, b_l).$$

Proof. Let $t = \mathrm{Tr}(C(a_k, b_l))$. We need to prove that

$$\mathrm{Tr}(a_j C_i(a_k, b_l)) = \begin{cases} 0, & i \neq j \\ t, & i = j \end{cases}.$$

Using that $\mathrm{Tr}(a_j(vu)) = \mathrm{Tr}(ua_jv)$, each monomial recovers the trace from the corresponding τ_i (i.e., $\mathrm{Tr}(a_j\tau_i(C)) = \mathrm{Tr}\, C(..., x_i = a_j, ...)$). When $i \neq j$, we get

$$\mathrm{Tr}(C(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, b_l)) = 0$$

(here we have replaced a_i with a_j and used that C is antisymmetric to conclude that the resulting trace is zero). When $i = j$, we get t. Moreover, plugging in the elementary matrices, we get that t is not uniformly zero. □

Definition 23.10: *The **Razmyslov polynomial** is*

$$Z_n(x_1, \ldots, x_{n^2}, y_0, \ldots, y_{n^2}, z) = \sum_i x_i z C_i(x_1, \ldots, x_{n^2}, y_0, \ldots, y_{n^2}).$$

Theorem 23.11: *The map $\mathrm{Mat}_n(k)^{2n^2+2} \to \mathrm{Mat}_n(k)$ sending $x_1, \ldots, x_{n^2}, y_0, \ldots, y_{n^2}, z$ to $Z_n(x_k, y_l, z)$ takes values in the scalar matrices and is not identically zero.*

Proof. The previous lemmas tell us that

$$Z_n(x_k, y_l, z) = \mathrm{Tr}(C(x_k, y_l))\,\mathrm{Tr}(z) I_n.$$

So we should find x_k, y_l for which $\mathrm{Tr}(C(x_k, y_l)) \neq 0$. Let $m_1, \ldots, m_{n^2}$ be the n^2 elementary matrices. Now if we have $m_a = e_{kl}$ and $m_{a+1} = e_{k'l'}$, let

$$y_a := e_{lk'},\ a = 1, \ldots, n^2 - 1$$

and saying $m_1 = e_{kl}$ and $m_{n^2} = e_{l'1}$, let

$$y_0 := e_{l'k}.$$

Now setting $x_k = m_k$, the monomial corresponding to the identity permutation evaluates to e_{11} while all the other monomials evaluate to 0.
Therefore, Z_n is nonzero here. □

23.5 Rowen's Theorem for semi-primitive algebras

To prove Rowen's Theorem, first we prove a version for semi-primitive PI algebras that uses the central polynomials above.

Proposition 23.12: *Let A be a semi-primitive PI algebra. Then every nonzero two-sided ideal meets the center.*

Proof. Let L be an irreducible A-module. Then for $D = \mathrm{End}_A(L)$ and $K = Z(\mathrm{End}_A(L))$, Kaplanksy's theorem, gives us a bound on $d(L) := \dim_D(L) \cdot \deg(D)$. The image $\bar{A}_L$ of the map $A \to \mathrm{End}(L)$ is isomorphic to $\mathrm{Mat}_m(D)$, and choosing a splitting field F of D, we get
$$\bar{A}_L \otimes_K F \cong \mathrm{Mat}_{d(L)}(F).$$
Let n be the maximal $d(L)$ such that $I \not\subset \mathrm{Ann}(L)$. Then we claim that our central polynomial $c = Z_n(x_k, y_l, z)$ for $z \in I$ lies in the center of A. We show that it will go to a central element in any irreducible L; this is enough because A is semi-primitive. If $d(L) > n$, then c acts by zero in L because z does. If $d(L) < n$, then c also maps to zero because Z_n is an identity in $\mathrm{Mat}_m(k)$ for $m < n$. If $d(L) = n$, z becomes a scalar matrix after extending scalars to F as above, so it lands in K.
The last thing is to show that $c \neq 0$. To do so, pick L with $d(L) = n, IL \neq 0$. Then I maps onto $\bar{A}_L = \mathrm{Mat}_m(D)$ so it suffices to show that Z_n is not an identity in $\mathrm{Mat}_m(D)$. But since identities are preserved by extension of scalars and Z_n is not an identity in $\mathrm{Mat}_n(F)$, Z_n is nonzero in $\mathrm{Mat}_m(D)$. □

23.6 Proof of Rowen's Theorem (for real)

Rowen's theorem follows from the above weaker version and

Theorem 23.13: *If R is a semi-prime PI algebra, then $R[t]$ is a semi-primitive PI algebra.*

Proof (of Theorem 23.5). If R is a semi-prime PI algebra then $R[t]$ is a PI algebra since extension of scalars to $R[t]$ preserves multi-linear identities, hence the PI property. Now if $I \subset R$ is a nonzero ideal, then $I[t] \subset R[t]$ will meet the center of $R[t]$ (by Theorem 23.13). But if $I[t]$ meets the center, then so does I. □

Theorem 23.13 will follow from the following.

Definition 23.14: *A **nil ideal** is an ideal consisting of nilpotent elements.*

Theorem 23.15 (Amitsur): *If R has no nonzero nil ideals, then $R[t]$ is semi-primitive.*

Proposition 23.16: *A semi-prime PI algebra contains no nil ideals.*

Proof (of Theorem 23.15). Let $J \subset R[t]$ be the Jacobson radical and suppose $p(t) = \sum a_i t^{r_i} \in J$ is a nonzero element of the Jacobson radical. WLOG, we can assume that the length of this sum is the minimal possible for a nonzero $p \in J$. Then the a_i must pairwise commute; otherwise, $[a_i, p]$ will be a shorter nonzero polynomial in J.
This implies that $1 + tp(t)$ is invertible in $R[t]$, and the coefficients of $(1 + tp(t))^{-1} \in R[[t]]$ lie in the commutative subring $S \subset R$ generated by the a_i. But for a commutative ring S, $1 + tp(t) \in S[t]$ is invertible iff all its coefficients are nilpotent: otherwise, we could find a maximal ideal $\mathfrak{m} \subset S$ such that $p \notin \mathfrak{m}[t]$ and p would be invertible over $(S/\mathfrak{m})[t]$, but nonconstant polynomials over fields cannot be invertible.
Therefore, each a_i is nilpotent. Then the set of all a_1 such that $q(t) = \sum a_i t^{r_i} \in J$ for some $a_2, \ldots, a_n$ is a nil ideal in R. So $J = 0$. □

Finally, we prove Proposition 23.16.

Lemma 23.17: *Suppose that a ring R satisfies the ascending chain termination condition for right annihilators. If R is semi-prime, then every nil left ideal is zero.*

Proof. Suppose I is a nil left ideal. WLOG we can assume that $I = Ra$ for some $a \in R$. Let $J = \mathrm{rAnn}(b) \subsetneq R$ for $b \in I, b = xa$ be maximal among right annihilators of (nonzero) elements in I. Then if $b^n \neq 0$, then
$$\mathrm{rAnn}(b^n) \supset \mathrm{rAnn}(b) \Rightarrow \mathrm{rAnn}(b^n) = \mathrm{rAnn}(b).$$
Hence $b^2 = 0$; otherwise for $n \geqslant 2$ we have $b^n \neq 0$ and $b^{n+1} = 0$, so $b \in \mathrm{rAnn}(b^n)$ and $b \in \mathrm{rAnn}(b)$.

We also claim $bRb = 0$. To see this, fix $y \in R$ and consider $c = yb \neq 0$. Pick n such that $c^n \neq 0, c^{n+1} = 0$. Then

$$c \in \mathrm{rAnn}(c^n),\ \mathrm{rAnn}(c^n) \supset \mathrm{rAnn}(b) \Rightarrow \mathrm{rAnn}(c^n) = \mathrm{rAnn}(b).$$

Then $c \in \mathrm{rAnn}(b) \Rightarrow byb = 0$. Thus, RbR is a nonzero nilpotent ideal, contradicting that R is semi-prime. □

Lemma 23.18: *A prime PI ring satisfies the ascending chain termination condition for right and left annihilators.*

Proof. Suppose $P(x_1, \dots, x_n) = \sum a_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ is a multilinear identity holding in R and $I_1 \subsetneq I_2 \subsetneq \cdots$ is an infinite ascending chain of left annihilators. Let $J_i = \mathrm{rAnn}(I_i)$, so $I_i \subset \mathrm{lAnn}(J_i)$.
Now evaluate P at $x_i \in I_i$. WLOG we can assume P is the smallest degree of an identity holding for such a choice of variable values. So for $y \in J_{n-1}$, $P(x_1, \dots, x_n)y = 0$. Also $x_{\sigma(1)} \cdots x_{\sigma(n)}y = 0$ when $\sigma(n) \neq n$ since $y \in J_{n-1}$, $x_{\sigma(n)} \in I_{\sigma(n)}$, and $I_{\sigma(n)}J_{n-1} = 0$.
So we can write $P = Q(x_1, \dots, x_{n-1})x_n +$ monomials not ending in x_n. The previous paragraph shows that for any $x_i \in I_i$,

$$Q(x_1, \dots, x_{n-1})I_n J_{n-1} = 0.$$

But R is prime, so $I_n J_{n-1} \neq 0$. Hence $Q(x_1, \dots, x_{n-1}) = 0$, which contradicts the degree minimality assumption. □

We've now proved Rowen's theorem if R is prime. If R is semi-prime, we need this last lemma:

Lemma 23.19: *A semi-prime ideal is an intersection of prime ideals.*

Proof. Let $I \subset R$ be a semi-prime ideal, i.e., $\bar{R} := R/I$ is semi-prime. Let $a \in \bar{R}$, $a \neq 0$. Since $\bar{R}$ is semi-prime, there exists $a_1 = a$ and $a_{i+1} = a_i x_i a_i$ such that $a_i \neq 0$ (construct a_i inductively, using that $(\bar{R}a_i\bar{R})^2 \neq 0$). Let J be a maximal ideal in $\bar{R}$ not containing a_i for any $i \geqslant 1$; this exists by Zorn's lemma.
Suppose J is not prime (i.e. ring $\bar{R}/J$ is not prime), so $x\bar{R}y \subset J$ for some $x, y \notin J$. Then for some i, $a_i \in \bar{R}x\bar{R} + J$ and $a_i \in \bar{R}y\bar{R} + J$ (use that if a_n lies in some ideal then a_k for every $k \geqslant n$ lies in the same ideal). But then

$$a_{i+1} \in \bar{R}x\bar{R}y\bar{R} + J = J$$

which contradicts our choice of J. So $a \notin J$ and J is prime.
Since we can find such J for each $a \neq 0$, then I is an intersection of these prime ideals. □

Therefore, a semi-prime ring R can be realized as a subring in the product of prime rings R/J. If R is a PI ring, then each R/J is such, so a nil ideal in R has zero image in R/J for all J, thus, it is zero. This finally proves Rowen's theorem.