Lecture 24: Gelfand-Kirillov Dimension

24.1 Growth of algebras and Gelfand-Kirillov dimension

Recall that we defined the growth of a (finitely generated) algebra as follows: pick a finite-dimensional space of generators V, which gives us a (surjective) homomorphism $TV \twoheadrightarrow A$ and induces a filtration of A by setting

$$A_{\leqslant n} = \operatorname{im}\left(\bigoplus_{i\leqslant n} V^{\otimes i}\right).$$

Let

$$d(n) := \dim(A_{\leqslant n}).$$

Then we saw that the order of growth was independent of V. Recall that A has

- subexponential growth if $d(n) < cn^{\alpha}$ for some c for all $\alpha > 1$
- exponential growth if $\limsup \sqrt[n]{d(n)} > 1$
- polynomial growth if there exists c, δ such that $d(n) \leqslant cn^{\delta}$.

Definition 24.1: *The* ***Gelfand-Kirillov dimension*** *of an algebra* A *is*

$$\inf\{\delta \mid \exists c, d(n) \leqslant cn^{\delta}\}.$$

That is, $\mathrm{GKdim}(A) \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$*, and this is well-defined since another function* d' *such that there exists* $a \geqslant 1$ *with*

$$d'(n/a) \leqslant d(n) \leqslant d'(an)$$

will lead to the same value.

Example 24.2: The GK dimension of $k[x_1, \ldots, x_n]$ is n.

Remark 24.3: There are similar definitions for finitely generated groups; if they have exponential growth, then they're called hyperbolic groups. (There is not much similarity in the methods and theorems though.)

Remark 24.4: If $A_{\leqslant n+1} = A_{\leqslant n}$, then $A = A_{\leqslant n}$ and $d(n)$ will eventually be $\dim A$. Hence, if $\dim A < \infty$, $\mathrm{GKdim}(A) = 0$. Otherwise, we'll always have $d(n) \geqslant n + 1$, so the GK dimension will be $\geqslant 1$.

Lemma 24.5:

a) *If* $\mathrm{GKdim}(A) < 1$*, then* A *is finite-dimensional so* $\mathrm{GKdim}(A) = 0$.

b) $\mathrm{GKdim}(A[t]) = \mathrm{GKdim}(A) + 1$

c) $\mathrm{GKdim}(A[a^{-1}]) = \mathrm{GKdim}(A)$ *if* a *is central and regular.*

Proof. a) If $d(n+1) = d(n)$ for some n, then $A_{\leqslant n+1} = A_{\leqslant n}$. So $A = A_{\leqslant n}$ and $d(n)$ will eventually be $\dim A$. Hence, if $\dim A < \infty$, $\mathrm{GKdim}(A) = 0$. Otherwise, we'll always have $d(n+1) \geqslant d(n) + 1$, so $d(n) \geqslant n$ and the GK dimension will be $\geqslant 1$.

b) Let $B = A[t]$ and $V_B = V_A \oplus kt$. Then $B_{\leqslant n} = t^n A_{\leqslant 0} \oplus t^{n-1} A_{\leqslant 1} \oplus \cdots \oplus A_{\leqslant n}$, so $\dim B_{\leqslant n} \leqslant (n+1) \dim A_{\leqslant n}$. Also, $\dim B_{\leqslant n} \geqslant n \dim A_{\leqslant n}$, so $\mathrm{GKdim}(A[t]) = \mathrm{GKdim}(A) + 1$.

c) Again, add a^{-1} to the space of generators, Then $\dim A_{\leqslant n} \leqslant \dim(B_{\leqslant n}) \leqslant \dim(A_{\leqslant 2n})$ because $B_{\leqslant n} \hookrightarrow A_{\leqslant 2n}$ via multiplication by a^n. So $\mathrm{GKdim}(A[a^{-1}]) = \mathrm{GKdim}(A)$.

□

Part b) implies that $\mathrm{GKdim}(k[x_1, \ldots, x_n]) = n$.

24.2 Warfield's Theorem

The GK dimension of a noncommutative ring can take any value $\geqslant 2$.

Theorem 24.6 (Warfield): *For any real* $\delta \geqslant 2$*, there exists an algebra with 2 generators whose GK dimension is* δ.

Proof. Part b) of Lemma 24.1 implies we only have to show this for $\delta \in (2, 3)$. We will construct a quotient of $k\langle x, y\rangle$ by monomials. Fix a monotonically increasing sequence γ_n, $n = 1, \ldots,$, let $I \subset k\langle x, y\rangle$ be the ideal spanned by the monomials of degree at least 3 in y and the monomials

$$x^i y x^j y x^k, \; j < \gamma_n, n = i + j + k.$$

The quotient $k\langle x, y\rangle / I$ is a graded algebra A; let A_n be the component of degree n. Then

$$\dim(A_n) = 1 + n + \binom{n + 2 - \gamma_n}{2}$$

where the $1 + n$ comes from monomials of degree $0, 1$ in y.
If we take $q \in (0, 1)$ and set $\gamma_n = n - [n^q]$, then $\mathrm{GKdim}(A) = \max(2, 2q + 1)$. Hence this gives you anything in $(2, 3)$. □

Remark 24.7: This doesn't happen for finitely presented monomial algebras. Notice that for every finitely generated algebra A, one can find a finitely generated monomial algebra $\bar{A}$ with the same growth function by setting $\bar{A}$ to be the associated graded for a filtration on A. But the same construction for A finitely presented does not imply that $\bar{A}$ is finitely presented.

24.3 Bergman gap theorem

The proof of this theorem is presented in [8] chapter 2.

Theorem 24.8 (Bergman gap): *There is no finitely generated algebra whose GK dimension is strictly between 1 and 2.*

Proof. The theorem follows from the below proposition. To reduce to a graded algebra generated in degree 1, we can reduce to $A = k\langle x_1, \dots, x_n\rangle / J$ where J is a monomial ideal. Then take the associated graded, first by total degree, then by lexicographical order. Then either $\dim A_d \geqslant d$, which implies that the GK dimension is at least 2, or there exists d with $\dim A_d < d$, which (by the proposition) implies that $\dim A_n$ is bounded by a constant and the GK dimension is at most 1. □

Proposition 24.9: *If A is a graded algebra generated in degree 1 and there exists d such that* $\dim A_d < d$*, then* $\operatorname{GKdim}(A) \leqslant 1$.

Proof. WLOG we can assume that all relations are monomial in degree d. To prove this, we define "allowed words", where a word is allowed iff all its subwords of length d are allowed. Let S be the set of allowed words of degree d and suppose $|S| \leqslant d$. Then the number of allowed words of degree N is bounded.
This reduces to

Lemma 24.10: *Assume that there at most d allowed words of length d. Then for $n \geqslant 2d$, every allowed word of length n has the form $w = w_1 w_2 w_3$ where w_2 is p-periodic for $p \leqslant d$, $|w_1|, |w_3| \leqslant d - p$, and $|w_2| \geqslant d + p$. (A finite word $x_1 \cdots x_n$ is p-periodic if $x_{i+p} = x_i$ when $i, i+p \in [1, \dots, n]$.)*

Proof. We induct on $|w|$. The base case is $|w| = 2d$. Such a word will have $d+1$ subwords of length d, but since there are only d distinct allowed words, at least two of these coincide and we have the desired periodicity.
Now we need the following:

Lemma 24.11: *If a periodic word with minimal period p contains two equal subwords of length $\geqslant p - 1$, then they are np letters apart.*

Proof. Extend the word to an infinite p-periodic word. Suppose the equal subwords are $x_{i+1} \cdots x_{i+r}$ and $x_{j+1} \cdots x_{j+r}$ with $r \geqslant p-1$. Then the subwords $x_i x_{i+1} \cdots x_{i+p-1}$ and $x_j x_{j+1} \cdots x_{j+p-1}$ are each a full period of the word x. Since $x_{i+q} = x_{j+q}$ for all $1 \leqslant q \leqslant r$, then $x_i = x_j$ also.
So x also has equal subwords $x_i \cdots x_{i+p-1}, x_j \cdots x_{j+p-1}$. Let the word have length m and $1 \leqslant \ell \leqslant m$, and let t be an integer such that $\ell + tp = i + s$ for $0 \leqslant s \leqslant p - 1$. Then

$$x_{\ell+(j-i)} = x_{\ell+(j-i)+tp} = x_{i+s+(j-i)} = x_{j+s} = x_{i+s} = x_{\ell+tp} = x_\ell$$

so x has period $j - i$. Thus x has period equal to the greatest common divisor of p, $j-i$ and the minimality of p implies that $p | j - i$ as desired. □

Now we finish the proof of the lemma. Write $w = x_1 w'$ and $w' = w_1' w_2' w_3'$. If $|w_1'| < d - p$, there's nothing to do. Otherwise, in $x_1 x_2 \cdots x_d$, find two coinciding length d words. These intersect w_2 by at least $p - 1$, so their intersections with w_2' differ by a shift by n and $p|n$. One of them ends at x_{2d-p} (or to the left) so it contains x_{d-p+1}. Hence $x_{d-p+1} = x_{d-p+1+n} = x_{d+1}$. □

This finishes the proof of the proposition. □

24.4 Ufnarovskii graph

Another way of working with allowed words is via the overlap graph, called the Ufnarovskii graph; the proof of the theorem can also be interpreted via the graph. Consider an oriented graph U whose vertices are allowed length d words and which has an edge between w_1 and w_2 iff w_2 is obtained from w_1 by removing the first letter and adding a letter at the end. Then paths of length $n - d$ correspond to allowed words of degree $n \geqslant d$.

The proof of the Bergman gap theorem can be restated as follows: if there are at most d allowed words of length d, show that U contains at most one oriented cycle. Then any path in the graph enters the cycle at most once, traverses the cycle, then leaves the cycle; this is the factorization $w = w_1 w_2 w_3$ in the lemma above (see [1, Section VI.4]).

24.5 Smoktunowicz and Berele theorems

We state without proof two related results:

Theorem 24.12 (A. Smoktunowicz): *The Gelfand-Kirillov dimension of a graded domain cannot fall within the open interval* $(2, 3)$.

Theorem 24.13 (Berele): *Finitely generated PI algebras have finite GK dimension.*

24.6 GK dimension of a module

Definition 24.14: *We can likewise define the Gelfand-Kirillov dimension of a finitely generated module over A by defining*

$$d_M(n) = \dim M_{\leqslant n}$$

where we pick generators $W \subset M$ *and* $M_{\leqslant n} = A_{\leqslant n} \cdot W$, *and setting*

$$\operatorname{GKdim}(M) = \inf\{\delta \mid \exists c, d_M(n) \leqslant c n^{\delta}\}.$$

Again, this is not dependent on the choices of W.

Definition 24.15: *We say that the GK dimension is **exact** for modules over an algebra A if for* $M \supset N$,

$$\operatorname{GKdim}(M) = \max(\operatorname{GKdim}(N), \operatorname{GKdim}(M/N)).$$

Example 24.16: GK dimension is exact for finitely generated modules over Noetherian PI algebras.

Suppose that A is an algebra with commutative associated graded (which also is then automatically finitely generated, hence Noetherian). Then the GK dimension is exact for (f.g. modules over) A, because

Proposition 24.17: *In this case,* $\operatorname{GKdim}(M)$ *is the dimension of the support of the* $\operatorname{gr}(A)$ *module* $\operatorname{gr}(M) = \bigoplus M_{\leqslant d}/M_{\leqslant d-1}$.

In fact, there is a closer relation between the commutative and noncommutative pictures. Let $\operatorname{gr} A = \bar{A}$. Given an increasing filtration on A such that $\bar{A}$ is commutative, let a good filtration on M be a filtration such that $M = \bigcup M_{\leqslant d}$, $\bigcap M_{\leqslant d} = 0$, $A_{\leqslant 1} M_{\leqslant n} \subset M_{\leqslant n+1}$, and $\operatorname{gr} M = \bar{M}$ is a finitely generated $\bar{A}$ module.

Lemma 24.18: *For* A, M *as above, the (set theoretic) support* $\operatorname{supp}(\operatorname{gr} M) \subset \operatorname{Spec}(\bar{A})$ *and does not depend on the choice of filtration. Moreover, the class of* $\bar{M}$ *in* $K\left(\bar{A}\text{-mod}_S\right)$ *(the Grothendieck group) is independent of the choice of the filtration, where* $\bar{A}\text{-mod}_S$ *is the category of finitely generated* $\bar{A}$*-modules with set-theoretic support contained in* S.

Remark 24.19: The expression "set-theoretic support" refers to thinking of finitely generated $\bar{A}$-modules as coherent sheaves on $\mathrm{Spec}(\bar{A})$. Closed subsets $S \subset \mathrm{Spec}(\bar{A})$ correspond to radical ideals $I_S \subset \bar{A}$, and M is set-theoretically supported on S iff every element of M is annihilated by some power of I_S. Note that being scheme-theoretically supported on S would instead mean that M is annihilated by I_S, which is stronger.

Proof (of lemma, sketch). Given two good filtrations $M_{\leqslant d}$ and $M'_{\leqslant d}$, find m such that

$$M_{\leqslant d-m} \subset M'_{\leqslant d} \subset M_{\leqslant d-m+1}.$$

Inducting on M, we can reduce to the situation when $m = 0$ and

$$M_{\leqslant d} \subset M'_{\leqslant d} \subset M_{\leqslant d+1}.$$

Let

$$N = \bigoplus M_{\leqslant d}/M'_{\leqslant d-1}, N' = \bigoplus M'_{\leqslant d}/M_{\leqslant d}.$$

Then there are short exact sequences

$$0 \to N' \to \bar{M} \to N \to 0$$
$$0 \to N \to \bar{M}' \to N' \to 0$$

which shows both statements. □

Remark 24.20: $\bar{M}$ is naturally graded, but the class of $\bar{M}$ in the Grothendieck group of graded $\bar{A}$-modules may depend on the choice of the filtration. This is because one can equip N, N' with a grading so that the first displayed SES is one of the graded modules, but the arrows in the second one will not agree with the grading.

24.7 Poincare series

Theorem 24.21 (Stephenson-Zhang): *If A is right (or left) Noetherian, it has subexponential growth.*

Lemma 24.22:
a) *A has exponential growth iff $a_n = \dim(A_n)$ has exponential growth iff $\limsup \sqrt[n]{a_n} > 1$.*
b) *For a sequence $a(n)$ of exponential growth, there exist $0 < r_1 < r_2 < \cdots$ such that*

$$a(r_k) < \sum_{i=1}^{k-1} a(r_k - r_i).$$

Proof. Having fixed $a(1), \ldots, a(m)$, there are infinitely many n such that

$$a(n) \geqslant \alpha^{r_i} a(n - r_i), i = 1, \ldots, m.$$

We can make the choice such that $\alpha^{r_k} > 2^k$. □

Proof (of theorem). Apply this to $a(n) = \dim A_n$. Inductively choose $x_i \in A_{r_i}$ such that $x_k \notin \sum_{i=1}^{k-1} x_i A_{k-i}$. □

Theorem 24.23: *Let $A = k \oplus \bigoplus_{i \geqslant 1} A_i$ be right (or left) Noetherian of right (or left) finite homological dimension. Then*

$$h(t) = \frac{1}{q(t)}, q(t) \in \mathbb{Z}[t]$$

where $h(t)$ is the Hilbert series and $q(t)$ is a polynomial whose roots are all roots of unity.

Proof. We must have

$$q(t) = \sum (-1)^i \dim \operatorname{Tor}_i^A(k,k) t^i$$

(i.e. the graded Euler characteristic of $\operatorname{Tor}^A(k,k)$).
All the roots z_i of $q(t)$ must have $|z_i| \geqslant 1$; otherwise, $\sum a_n t^n$ has radius of convergence < 1 and A has exponential growth. But $\prod z_i = 1$ because their product will be 1/the leading coefficient of $q(t)$. So all the roots satisfy $|z_i| = 1$. Since they are roots of a polynomial in $\mathbb{Z}[t]$, they are roots of unity. □

Conjecture 24.24 (Polishchuk-Positselski): *The Hilbert series of a Koszul algebra is rational. Moreover, if both A and $A^!$ have finite GK dimension, then they have the Hilbert series of a symmetric tensor exterior.*

Conjecture 24.25 (Anick): *Assume A is right Noetherian. If both* $\operatorname{GKdim}(A)$ *and* $\operatorname{hdim}(A)$ *are finite, then the Hilbert series of A equals that of the symmetric algebra.*

18.706 Noncommutative Algebra
Spring 2023