Lecture 25: Final Class: Noncommutative Geometry

Recall that commutative algebra is closely related to algebraic geometry. A commutative ring R corresponds to the affine scheme $\operatorname{Spec} R$, and modules over R correspond to sheaves on $\operatorname{Spec} R$. In algebraic geometry these concepts are extended to more general non-affine schemes, while also creating powerful geometric intuition and techniques that have had a strong impact on commutative algebra.

Noncommutative geometry is an area that grew out of attempts to tie noncommutative algebra to geometry in a similar way. This has not led to as comprehensive a theory as exists in the commutative case. However, it did lead to emergence of a number of different directions, some leading to impressive results.

In this lecture we will briefly survey some of these directions. Our list is by no means complete; for example we don't discuss the direction involving tools from functional analysis (C^*-algebras) developed by A. Connes et al.

25.1 Representation varieties

Let R be a finitely generated commutative ring over $k = \bar{k}$. Then k-points of $\operatorname{Spec}(R)$ correspond to the homomorphisms of k-algebras $\operatorname{Hom}(R, k)$ i.e. to *one-dimensional* representations of the algebra R.

Note that every simple module over a finitely generated commutative algebra R is one-dimensional (Hilbert's Nullstellensatz). If R is instead a finitely generated noncommutative algebra over k, it is natural to consider the space of *all* finite-dimensional representations of R. Let us describe this space (to be denoted $\operatorname{Rep}(R)$).

First of all note that $\operatorname{Rep}(R) = \bigsqcup_{n \in \mathbb{Z}_{\geqslant 0}} \operatorname{Rep}_n(R)$, where $\operatorname{Rep}_n(R)$ is the space of n-dimensional representations of R.

Every element of $\operatorname{Rep}_n(R)$ corresponds to a homomorphism $\varphi\colon R \to \operatorname{Mat}_n(k)$ i.e. $\varphi \in \operatorname{Hom}(R, \operatorname{Mat}_n(k))$. Two homomorphisms φ_1, φ_2 define isomorphic representations iff they lie in the same orbit of GL_n, acting naturally on the space $\operatorname{Hom}(R, \operatorname{Mat}_n(k))$ (via its action on $\operatorname{Mat}_n(k)$). Since R is finitely generated, $R = k\langle x_1, \ldots, x_m \rangle / I$, and

$$\operatorname{Hom}(R, \operatorname{Mat}_n(k)) \subset (\operatorname{Mat}_n)^m = k^{n^2 m} = \mathbb{A}^{n^2 m}$$

is a subset of the affine variety $(\operatorname{Mat}_n)^m$ cut out by polynomial equations. We can consider this subset as an *algebraic subvariety* of $\mathbb{A}^{n^2 m}$. We see that $\operatorname{Rep}_n(R) = \operatorname{Hom}(R, \operatorname{Mat}_n(k))/\mathrm{GL}_n(k)$ is the quotient of the algebraic variety $\operatorname{Hom}(R, \operatorname{Mat}_n(k))$ by the action of the algebraic group $\mathrm{GL}_n(k)$. Space $\operatorname{Rep}_n(R)$ is an example of an *algebraic stack*. This is a replacement for $\operatorname{Spec}(R)$.

Preprojective algebras are examples of explicit algebras with interesting representation varieties $\operatorname{Rep} R$.

Let Q be an oriented quiver and let $\overline{Q}$ be the corresponding double quiver. For an edge e of Q we will denote by e_+, e_- the corresponding edges of $\overline{Q}$. Let $P(Q)$ be the quiver algebra of $\overline{Q}$ modulo the relation

$$\sum_e e_- e_+ - \sum_e e_+ e_- = 0. \tag{3}$$

For example, let Q be a cyclic quiver consisting of n vertices labeled by the elements of $\mathbb{Z}/n\mathbb{Z}$ (vertices $[i]$, $[i+1]$ are connected by the edge). Quiver $\overline{Q}$ has vertices labeled by $\mathbb{Z}/n\mathbb{Z}$, edges of this quiver are $[i] \leftarrow [i+1]$ and $[i+1] \leftarrow [i]$, $[i] \in \mathbb{Z}/n\mathbb{Z}$.

Pick $\zeta \in k$ of order n and consider the action $\mathbb{Z}/n\mathbb{Z} \curvearrowright k[x,y]$ given by $[1]\cdot x = \zeta x$, $[1]\cdot y = \zeta^{-1}y$. We have $(\mathbb{Z}/n\mathbb{Z})\#k[x,y] \xrightarrow{\sim} P(Q)$, the isomorphism is given by:

$$1 \otimes x \mapsto \sum_{[i]\in\mathbb{Z}/n\mathbb{Z}} e_{[i+1]\leftarrow[i]},\ 1 \otimes y \mapsto \sum_{[i]\in\mathbb{Z}/n\mathbb{Z}} e_{[i]\leftarrow[i+1]},\ [1]\otimes 1 \mapsto \sum_{[i]\in\mathbb{Z}/n\mathbb{Z}} \zeta^i e_{[i]}.$$

The isomorphism above induces the equivalence between the categories of $P(Q)$ and $(\mathbb{Z}/n\mathbb{Z})\#k[x,y]$-modules. Let us describe this equivalence explicitly. Module $(M_{[i]})_{[i]\in\mathbb{Z}/n\mathbb{Z}}$ over $P(Q)$ goes to $M := \bigoplus_{[i]\in\mathbb{Z}/n\mathbb{Z}} M_{[i]}$, where the action of $[1] \in \mathbb{Z}/n\mathbb{Z}$ on $M_{[i]}$ is given by ζ^i and the action of $x\colon M_{[i]} \to M_{[i+1]}$ is given by $e_{[i+1]\leftarrow[i]}$, the action of $y\colon M_{[i+1]} \to M_{[i]}$ is given by $e_{[i]\leftarrow[i+1]}$. The condition $\sum_{[i]\in\mathbb{Z}/n\mathbb{Z}} e_{[i]\leftarrow[i+1]}e_{[i+1]\leftarrow[i]} = \sum_{[i]\in\mathbb{Z}/n\mathbb{Z}} e_{[i+1]\leftarrow[i]}e_{[i]\leftarrow[i+1]}$ precisely corresponds to the fact that x and y commute.

This example can be generalized as follows. Recall that finite subgroups Γ in $\mathrm{SL}(2,k)$ correspond to simply laced Dynkin graphs D (this is known as McKay correspondence, see [15]). Let $\widehat{D}$ be the affine Dynkin graph; the vertices of $\widehat{D}$ are in bijection with irreps of Γ (see [15]). Then (see [7])

$$P(\widehat{D}) \sim \Gamma\#k[x,y]$$

where the $\sim$ is Morita equivalence. It sends a $\Gamma\#k[x,y]$-module M to $\bigoplus_v M_v$, where $M_v = [M : \rho_v] = \mathrm{Hom}_\Gamma(\rho_v, M)$ and ρ_v is the irreducible representation of Γ corresponding to the vertex v.

Remark 25.1: Note that the algebras $P(\widehat{D})$, $\Gamma\#k[x,y]$ are not isomorphic in general (they are isomorphic for $\Gamma = \mathbb{Z}/n\mathbb{Z}$).

Let us describe the representation variety of the algebra $R = P(Q)$. Let us first of all recall that $P(Q)$ is a certain quotient of the path algebra of the quiver $\overline{Q}$ so every representation of $P(Q)$ can be considered as a representation of the quiver $\overline{Q}$ such that (3) holds. So, $\mathrm{Rep}(P(Q)) = \bigsqcup_{d_v\in\mathbb{Z}_{\geqslant 0}} \mathrm{Rep}_{d_v}(P(Q))$, where $\mathrm{Rep}_{d_v}(P(Q))$ is the space of representations (M_v) of $\overline{Q}$ such that (3) holds and $\dim M_v = d_v$ (considered up to an isomorphism). Explicitly, let

$$\widetilde{\mathrm{Rep}}_{d_v}(P(Q)) \subset \prod_{e\colon v\to v'} \mathrm{Mat}_{d_v,d_{v'}} \times \mathrm{Mat}_{d_{v'},d_v} \tag{4}$$

be the subvariety consisting of collections of maps such that (3) holds. Then

$$\mathrm{Rep}_{d_v}(P(Q)) = \widetilde{\mathrm{Rep}}_{d_v}(P(Q))/\prod_v \mathrm{GL}(d_v).$$

Note now that the RHS of (4) is a *symplectic vector space* (we identify $\mathrm{Mat}_{d_v,d_{v'}} \times \mathrm{Mat}_{d_{v'},d_v}$ with $T^*\,\mathrm{Mat}_{d_v,d_{v'}}$ via the trace form), and $G = \prod \mathrm{GL}(d_v)$ acts on it preserving the symplectic structure; the equations (3) are zeroes of the moment map $\mu\colon \prod_{e\colon v\to v'} T^*\,\mathrm{Mat}_{d_v,d_{v'}} \to \prod_v \mathrm{Mat}_{d_v}$ for the $\prod_v \mathrm{GL}_{d_v}$ action. So, $\mathrm{Rep}_{d_v}(P(Q))$ is obtained from the RHS by Hamiltonian reduction i.e.

$$\mathrm{Rep}_{d_v}(P(Q)) = \mu^{-1}(0)/\prod_v \mathrm{GL}_{d_v}$$

is the *Hamiltonian reduction* of the symplectic vector space $\prod_{e\colon v\to v'} T^*\,\mathrm{Mat}_{d_v,d_{v'}}$ by $\prod_v \mathrm{GL}_{d_v}$.

Remark 25.2:
Ginzburg introduced a notion of a *quiver with potential* (see [8, Section 4.2]). Potential is an *element* of the vector subspace F_{cyc} of the quiver algebra $F = kQ$ of a quiver Q generated by *cyclic* paths.
For any edge $e \in D$ there exists a map $\frac{\partial}{\partial e}\colon F_{\mathrm{cyc}} \to F$ defined as follows: given a cyclic path $\Phi = e_{i_1}e_{i_2}\dots e_{i_r}$, we put

$$\frac{\partial \Phi}{\partial e} := \sum_{\{s \mid e_{i_s} = e\}} e_{i_{s+1}}e_{i_{s+2}}\dots e_{i_r}e_{i_1}e_{i_2}\dots e_{i_{s-1}}.$$

One can then consider the quotient algebra $\mathfrak{A}(Q,\Phi) := F/(\frac{\partial\Phi}{\partial e})_{e\in Q}$, where $(\frac{\partial\Phi}{\partial e})_{e\in D}$ is the two-sided ideal generated by the elements $\frac{\partial\Phi}{\partial e} \in F$ (see [8, Equation (4.2.1)]).
The variety $\widetilde{\mathrm{Rep}}_{d_v}\mathfrak{A}(Q,\Phi)$ can be described as follows. Consider the space $\widetilde{\mathrm{Rep}}_{d_v}(F)$ (that is just a vector space), element Φ defines a map $\widehat{\Phi}\colon \widetilde{\mathrm{Rep}}_{d_v}(F) \to \prod_v \mathrm{Mat}_{d_v}$ sending a representation ρ to $\rho(\Phi)$ (recall that $\Phi \in F_{\mathrm{cyc}}$). We obtain a functional $\phi := \mathrm{tr}\,\widehat{\Phi}\colon \widetilde{\mathrm{Rep}}_{d_v}(F) \to \mathbb{C}$. One can show that the variety $\widetilde{\mathrm{Rep}}_{d_v}\mathfrak{A}(Q,\Phi)$ is the *critical locus* of $\mathrm{tr}\,\widehat{\Phi}$ (cf. [8, Section 2.3]).
Let us describe how to obtain preprojective algebras via algebras with potentials (see [8, Example 4.3.3]). Consider the quiver $\overline{Q}^{\mathrm{loop}}$ obtained from $\overline{Q}$ by attaching an additional edge loop, t_v, for every vertex v. We can identify the quiver algebra $k\overline{Q}^{\mathrm{loop}}$ with $k\overline{Q} * k[t]$, where $*$ corresponds to the free product of associative k-algebras (this isomorphism sends t to $\sum_v t_v$). Consider the potential

$$\Phi := \sum_v t_v \cdot \sum_e (e_+e_- - e_-e_+) = t\sum_e (e_+e_- - e_-e_+).$$

We then have $\mathfrak{A}(k\overline{Q}^{\mathrm{loop}},\Phi) = P(Q)[t]$ (see [8, Equation (4.3.4)]), so $\widetilde{\mathrm{Rep}}_{d_v}\mathfrak{A}(k\overline{Q}^{\mathrm{loop}},\Phi) = \widetilde{Rep}_{d_v}(P(Q)) \times \prod_v \mathrm{Mat}_{d_v}$. The potential ϕ that we construct above is given by the formula:

$$\prod_{e\colon v\to v'} T^*\,\mathrm{Mat}_{d_v,d_{v'}} \times \prod_v \mathrm{Mat}_{d_v} \ni (v,\xi) \mapsto \mathrm{tr}(\mu(v)\xi) \in \mathbb{C}$$

in this case.

25.2 Weyl algebra and deformations

Many interesting algebras have no nonzero finite-dimensional representations (in particular, $\mathrm{Rep}\,R = \{0\}$ is not interesting in this case). For example, the Weyl algebra $W = \mathbb{C}\langle x,y\rangle/\langle xy - yx - 1\rangle$ doesn't. You can see this by noting that $\mathrm{tr}(xy - yx) = 0$ (on every finite-dimensional representation), while $\mathrm{tr}(1_n) = n$.

We can study noncommutative geometry by deforming from the commutative case, this procedure is also called deformation quantization. Consider

$$W_\hbar = \mathbb{C}[\hbar]\langle x,y\rangle/\langle xy - yx - \hbar\rangle.$$

When we take $\hbar = 1$, we recover W, and when we take $\hbar = 0$, we get $\mathbb{C}[x,y]$, which is commutative.

Other examples of deformations:

If X is a smooth affine algebraic variety over a field k, we can consider $\mathrm{Diff}_\hbar(X)$, the asymptotic differential operators. This is W when $X = \mathbb{A}^1$. If $\hbar = 0$, we get $\mathcal{O}(T^*X) = \mathrm{Sym}_{\mathcal{O}(X)}(\mathrm{Der}(\mathcal{O}(X)))$.

If $\mathfrak{g}$ is a Lie algebra, let $U_\hbar(\mathfrak{g}) = k\langle\mathfrak{g}\rangle/xy - yx = \hbar x$. If $\mathfrak{g} = \mathfrak{gl}_n$, then define

$$\bar{U_\hbar}(\mathfrak{g}) = U_\hbar \otimes_{Z(U_\hbar(\mathfrak{g}))} k[\hbar]$$

so

$$\bar{U_0} = \mathcal{O}(\mathcal{N})$$

where $\mathcal{N}$ is the nilpotent matrices.

There's also the spherical rational double affine Hecke algebra (DAHA) $A_\hbar$, also called rational Cherednik algebra, where $A_0 = \mathcal{O}((\mathbb{A}^2)^n/S_n)$.

How can we define deformations for non-affine varieties? Previously we deformed the algebra of functions on X, now we need to deform the structure sheaf of X. There is no obvious way to make a deformation into a sheaf. Here are some ways:

a) We can work with a formal parameter. If A is flat over $k[[\hbar]]$ and complete in the $\hbar$-topology, let $A_0 = A/\hbar$, which is commutative. Exercise: a) if $\bar{a} = a \pmod{\hbar}$ is invertible, then so is a. b) If $U \subset \mathrm{Spec}(A)$ is open, then $\{a \in A | \bar{a}|_U \text{ is invertible}\}$ is a localizing class. c) Localizations form a Zariski sheaf on $\mathrm{Spec}(A)$.

b) $\mathrm{Diff}(X)$ can be made into a sheaf on T^*X with conical topology ($U \subset \mathrm{Spec}(T^*X)$ is open in a conical topology if it is open in Zariski topology and invariant under dilation).

c) In characteristic p, for $W = k\langle x, y\rangle / xy - yx = 1$ has $x^p, y^p \in Z(W)$. Hence, W is a sheaf on $\mathrm{Spec}\, k[x^p, y^p]$.

25.3 $\mathrm{Coh}(X)$ and $D^\flat(\mathrm{Coh}(X))$

The previous two subsections described approaches closely tied to the usual commutative algebraic geometry. The next relies on it as a source of motivation for conjectures rather than trying to relate a noncommutative structure to a specific commutative ring or variety.

An algebraic variety X can be studied via the category $\mathrm{Coh}(X)$ on $D^\flat(\mathrm{Coh}(X))$.

If $X = \mathrm{Spec}(R)$ is affine, then $\mathrm{Coh}(X) = R$-mod. If Y is projective over k and $Y = \mathrm{Proj}(A)$, $A = \bigoplus_{n \geq 0} A_n$, $A_0 = k$, then $\mathrm{Coh}(Y)$ is the Serre quotient of graded finitely generated A-modules by graded finite-dimensional A-modules.

Theorem 25.3 (Serre): *Let X be an algebraic variety over a field k. Then X is smooth iff $\mathrm{Coh}(X)$ has finite homological dimension, i.e. $\mathrm{Ext}^n(F, G) = 0$ for all $n > d$ and all $F, G \in \mathrm{Coh}(X)$.*

It is also known that X is projective iff $\mathrm{Ext}^n(F, G)$ is finite-dimensional for all n, $F, G \in \mathrm{Coh}(X)$.

Let X be a smooth affine variety. Then Ω^i_X, the i-forms, is $\mathrm{HH}_i(\mathcal{O}(X))$. Recall that HH^i and HH_i are Morita invariant. They can also be defined starting from a category: $\mathrm{HH}^i = \mathrm{Ext}^i(\mathrm{Id}, \mathrm{Id})$ where Id is the identity functor, while HH_i relates to the tensor of bimodules.

For X smooth and projective,

$$\mathrm{HH}_i(\mathrm{Coh}(X)) = \bigoplus_{q-p=i} H^p(X, \Omega^p) \simeq H^i_{\mathrm{dR}}(X)$$

where the last equivalence is from the Hodge theorem, and the first one is known as the Hochschild-Kostant-Rosenberg isomorphism.

To recover H^*_{dR} for nonprojective X, we can use cyclic homology. The bar complex for HH_* has cyclic symmetry:

$$C : a_0 \otimes \cdots a_n \longrightarrow (-1)^n a_n \otimes a_0 \cdots \otimes a_{n-1}.$$

Then $\mathrm{Bar}/(C - \mathrm{Id})\mathrm{Bar}$ inherits the differential from the bar complex. Its cohomology is

$$\mathrm{HC}_n(A) = \Omega^n / d\Omega^{n-1} \oplus \bigoplus_{i \geq 1} H^{n-2i}_{\mathrm{dR}}(X)$$

and

$$\mathrm{HC}^{\mathrm{per}}_n = \lim_{\rightarrow} \mathrm{HC}_{n+2i} = \bigoplus_{i=-\infty}^{\infty} H^{n+2i}_{\mathrm{dR}}(X).$$

For a smooth projective dimension n variety X we have Serre duality:

$$\mathrm{Ext}^i(F, G) \simeq \mathrm{Ext}^{n-i}(G, F \otimes K_\times)^*.$$

Definition 25.4 (Bondal-Kapranov): *Let C be a finite type k-linear triangulated category. (Finite type means* $\dim_k \mathrm{Ext}^*(A,B) < \infty \, \forall A, B$.*) For example, $C = D^b(A\text{-}mod)$ where A is finite-dimensional and of finite homological dimension.*
A Serre functor is a functor $S : C \to C$ and an isomorphism $\mathrm{Hom}(A,B) \simeq \mathrm{Hom}(B, S(A))^*$. *The Yoneda lemma implies that if S exists, it is unique.*

Example 25.5: For $C = D^b(\mathrm{Coh}(X))$, $S : F \mapsto F \otimes K_x[n]$ is a Serre functor.

The Hodge theorem can be restated as a claim that a spectral sequence $\mathrm{HH}_*(\mathrm{Coh}(X)) \implies \mathrm{HC}_*^{\mathrm{per}}(\mathrm{Coh}(X))$ degenerates for smooth projective X. The following striking generalization was proposed (in a slightly different form) by Kontsevich and Soibelman (see [11]).

Conjecture 25.6: *The above spectral sequence degenerates for any dg-category of finite type over k.*

This was partly proved by Dmitry Kaledin and Akhil Mathew, see [10], [14].

25.4 Artin-Schelter regular algebras and noncommutative projective geometry

A projective variety $X \subset \mathbb{P}_k^n$ is determined by its homogeneous coordinate ring A, a graded commutative ring such that $X = Proj(A)$. An important invariant of X is the abelian category $Coh(X)$ of coherent sheaves on X. It can be realized as a Serre quotient $A - mod_{fg}^{gr}/A - mod_{fd}^{gr}$ where $A - mod_{fg}^{gr}$ is the category of finitely generated graded A-modules and $A - mod_{fd}^{gr}$ is the Serre subcategory of finite dimensional graded modules.

One can study noncommutative graded ring that share basic features with commutative ones, thinking of them as homogeneous coordinate rings of (yet to be defined) noncommutative projective varieties. Some beautiful results in that direction were obtained by Artin, Schelter and others in 1990's.

So consider a nonnegatively graded algebras over a field $A = \oplus A_n$, $A_0 = k$. Assuming A is Noetherian, the category $A - mod_{fg}^{gr}$ is abelian, so one can consider $A - mod_{fg}^{gr}/A - mod_{fd}^{gr}$, the category of coherent sheaves on the purported noncommutative $Proj$ of A.

One defines a *point module* over A as a cyclic graded module with Poincare series $1/(1-t)$. In the case when A is commutative and generated by A_1 point modules are in bijection with points of $X = Proj(A)$.

Several important classification results are achieved by considering point modules and a usual (commutative) algebraic variety arising as the moduli space of point modules.

We briefly mention a sample classification problem. Recall that a commutative A as above has finite homological dimension (equivalently, is regular) iff it is a polynomial algebra. Assuming it is generated by A_1, we get the homogeneous coordinate ring of $X = \mathbb{P}_k^n$. Generalization of this simplest projective variety leads to the following definition.

An algebra A as above is called *Artin-Schelter regular* if it has finite homological dimension d, a finite GK dimension and $Ext_A^i(k, A) = 0$ for $i \neq d$, while $Ext_A^d(k, A)$ is one dimensional.

The work of Artin-Schelter and Artin-Tate-van den Bergh achieved classification of AS regular algebras of dimensions two and three (noncommutative projective lines and planes). We will not present the answer, but mention that it involves beautiful and rather elementary algebro-geometric data, such as an elliptic curve with an automorphism (see [17] and references therein), arising in the process of classification of point modules over A.

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