HOMEWORK 1 FOR 18.706, SPRING 2023

(1) Show that the converse to Schur Lemma is false by constructing a three dimensional algebra A over $\mathbb{C}$ and a two dimensional module M over A, such that M is reducible but $End_A(M) \cong \mathbb{C}$.

Can a module whose endomorphisms form a division ring be decomposable?

(2) Let R be the subring in $Mat_2(\mathbb{R})$ given by $R = \{(a_{ij}) \mid a_{21} = 0,\, a_{22} \in \mathbb{Q}\}$. Show that R is left Artinian and Noetherian but it is neither right Artinian nor right Noetherian.

Deduce that R is not isomorphic to R^{op}.

(3) This problem illustrates that a finite dimensional algebra may have indecomposable modules of an arbitrarily large dimension.

Let k be a field and $I \subset k[x,y]$ be the ideal generated by x, y. Let $A = k[x,y]/I^2$. Show that $M_n = I^n/I^{n+2}$ is an indecomposable A-module.

Construct an example of an infinite dimensional indecomposable A-module.

(4) Describe the socle and the co-socle filtration of the free rank one module for the following rings.
   (a) $R = \mathbb{Z}/72$.
   (b) $R = k[D_4]$, where k is a field of characteristic two, and D_4 denotes the dihedral group of order 8 (the group of symmetries of the square).

(5) Let Q be a quiver, i.e. a finite oriented graph. Let $A(Q)$ be the path algebra of Q over a field k, i.e. the algebra whose basis is formed by paths in Q (compatible with orientations, and including paths of length 0 from a vertex to itself), and multiplication is concatenation of paths (if the paths cannot be concatenated, the product is zero).
   (a) Represent the algebra of upper triangular matrices as $A(Q)$.
   (b) Show that $A(Q)$ is finite dimensional iff Q is acyclic, i.e. has no oriented cycles.
   (c) For any acyclic Q, decompose $A(Q)$ (as a left module) in a direct sum of indecomposable modules, and classify the simple $A(Q)$-modules.

(6) This problem provides examples showing that the conclusion of the Krull-Schmidt Theorem does not hold without the finiteness assumption on the module.
   (a) Let R be a Dedekind domain[1] which is not a principal ideal domain (e.g. $R = \mathbb{Z}[\sqrt{-5}]$ or $R = \mathbb{C}[x,y]/(y^2 - x^3 - 1)$). Show that the conclusion of Krull-Schmidt Theorem does not hold for finitely generated projective R modules.

[1]Recall that this means that R is a Noetherian (commutative) domain where every ideal is a product of prime ideals. Rings of integers in number fields provide important examples of Dedekind domains. Another class of examples comes from coordinates rings of smooth affine algebraic curves over a field.

[Hint: you can use without proof that $I \oplus J \cong R \oplus IJ$ for nonzero ideals $I, J \subset R$ and that isomorphism classes of nonzero ideals in R form a group under the operation $I \cdot J = IJ$.]

(b) Let A be the algebra of smooth real functions on the real line, such that $a(x+1) = a(x)$. Let M be the A-module of smooth functions on the line such that $b(x+1) = -b(x)$.
Show that M is indecomposable and not isomorphic to A, and that $M \oplus M \cong A \oplus A$ as a left A-module. Thus the conclusion of Krull-Schmidt theorem does not hold in this case.

18.706 Noncommutative Algebra
Spring 2023