HOMEWORK 2 FOR 18.706, SPRING 2023

(1) Is it true that every indecomposable module over an Artinian ring is a quotient of an indecomposable projective module? Prove or give a counterexample.

(2) Show a map from a projectve $\mathbb{Z}$-module to $M = \mathbb{Z}/p\mathbb{Z}$ can not be an essential surjection.

(3) (a) Let R be a ring and $I \subset R$ a (2-sided) nilpotent ideal. It was proved in class that every idempotent $e \in R/I$ admits a lifting to an idempotent $\tilde{e} \in R$. Prove that such a lifting is unique up to conjugation by an element in $1 + I$.

(b) Let R be an Artinian ring. Prove that the set of conjugacy classes of idempotents in R is finite and give a formula for the cardinality of that finite set in terms of dimensions of irreducible representations of R as vector spaces over their respective skew fields of endomorphisms.

(c) Show that the following rings have no idempotents except for the unit element and zero.

(i) $k[G]$ where k is a characterisitc p field and G is a finite group of order p^n.
[Hint: you can use without proof that the only irreducible representation of G over k is the trivial representation.]

(ii) (optional[1]) $k[G]$ where k is any field and G is a torsion free group (i.e. $g^n \neq 1$ if $g \in G$, $g \neq 1$).

(4) Let R be an Artinian ring. For a module M let $C(M)$ be the set of cyclic elements in M, i.e. $m \in C(M)$ iff $Rm = M$. Let M be an R module such that $C(M) \neq \emptyset$.

Show that the following are equivalent.
i) The complement $M \setminus C(M)$ is a submodule.
ii) M is a quotient of an indecomposable projective R-module.

(5) An Artinian ring is called self-injective if the free module is injective. A finite dimensional algebra A over a field k is called Frobenius if there exists a linear functional τ on A such that the bilinear pairing $A \times A \to k$, $(x, y) \mapsto \tau(xy)$ is non-degenerate.[2]

(a) Prove that a Frobenius algebra is self-injective.

(b) Let A be a finite dimensional algebra over a field. Assume that for every simple A-module L the multiplicity of L in the co-socle of A viewed as a left A-module equals the multiplicity of L^* in the socle of A viewed as a right A-module. Show that A is self-injective.

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[1] In fact, it is not recommended to seriously attempt this problem.

[2] It is easy to see that the group algebra $k[G]$ of a finite group G and exterior algebra of a finite dimensional vector space are examples of Frobenius algebras. Another class of examples is provided by cohomology of compact oriented manifolds, this follows from Poincare duality.

(Here we use that for a left A-module M the dual vector space $M^* = Hom(M, k)$ carries a right A-module structure, the action is given by the adjoint operators).

(6) Let Γ be a finite group acting on a ring R by automorphisms. Then the *smash product*[3] $\Gamma \# R$ or $\Gamma \ltimes R$ is the abelian group $\bigoplus_{\gamma\in\Gamma} R$ with multiplication given by: $(r_{\gamma_1})(r'_{\gamma_2}) = r\gamma_1(r')_{\gamma_1\gamma_2}$ in the self-explanatory notation.

Suppose that R is simple and $|G|$ is invertible in R.

Suppose also that no nontrivial element $\gamma \in \Gamma$ acts by an inner automorphism of R.

Prove that the rings $\Gamma \# R$-modules and R^Γ are Morita equivalent.

[Hint: reduce to showing that $e = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \gamma$ generates the unit two-sided ideal in $S = \Gamma \# R$. Now consider S as a bimodule over R and observe that the summands in the decomposition $S = \bigoplus_{\gamma\in\Gamma} R\gamma$ are pairwise nonisomorphic simple modules over $R \otimes_{\mathbb{Z}} R^{op}$. Now recall that the only submodules in a direct sum of nonisomorphic simple modules are direct sums of several of these summands.]

[3] I prefer the notation which can be described as "semi-tensor product" but I don't know how to reproduce it in LaTeX!

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