HOMEWORK 3 FOR 18.706, SPRING 2023

(1) Let $R = \Gamma\#\mathbb{C}[x]$ where $\Gamma = \mathbb{Z}/2\mathbb{Z}$, with nontrivial element of Γ acting by $x \mapsto -x$ (see problem 6 of pset 2 for notation).
   (a) Identify R with the subring in $Mat_2(\mathbb{C}[x])$ consisting of matrices (P_{ij}) such that $P_{ij}(-x) = (-1)^{i+j} P_{ij}(x)$.
   (b) Compute the center and the cocenter of R and check that the latter is not a free module over the former. Conclude that R is not Morita equivalent to a commutative ring.

(2) Let R be a ring and F be the forgetful functor from the category of R-modules to abelian groups. Describe $End(F)$.
   More generally, let $A \to R$ be a ring homomorphism. Describe the endomorphism ring of the pull-back functor $R - mod \to A - mod$.

(3) For each of the following functors determine existence of a left adjoint, of a right adjoint and describe the existing adjoint functors.
   (a) Let $Q = (\bullet \longrightarrow \bullet)$ be the quiver with two vertices and one arrow between them. Let $\mathcal{B}$ be the category of representations of Q over a fixed field, $\mathcal{A}$ be the full subcategory consisting of such representations that the map between the two vector spaces is injective, and $F : \mathcal{A} \to \mathcal{B}$ be the embedding.
   (b) A graded commutative (or *super-commutative*) ring is a $\mathbb{Z}/2\mathbb{Z}$ graded ring $R = R_0 \oplus R_1$ such that $xy = -yx$ for $x, y \in R_1$ and $xy = yx$ for $x \in R_0$, $y \in R$.
   Let $\mathcal{A}$ be the category of vector spaces over a field k not of characteristic two, $\mathcal{B}$ be the category of graded commutative k-algebras and let F send a vector space to its exterior algebra.

(4) In the proof of an explicit characterization of Morita equivalent rings we checked that a projective module P is finitely generated iff the functor $C_T : T \mapsto Hom(P, T)$ preserves arbitrary coproducts. This exercise shows this is not necessarily true when P is not projective.
   (a) Show that a module is finitely generated iff the union of any totally ordered (under inclusion) family of proper submodules is a proper submodule.
   (b) Show that for a module M the functor C_T preserves coproducts iff the union of any ordered sequence of proper submodules $M_1 \subseteq M_2 \subseteq \cdots$ is a proper submodule.
   [Hint: if a homomorphism $f : M \to \oplus N_\alpha$ does not land in the sum of a finite subset of N_α, one can choose $\alpha_1, \alpha_2, \dots$ so that the composition $M \to \oplus N_\alpha \to N_{\alpha_i}$ is nonzero for all i. Then consider $\bar{f} : M \to \bigoplus N_{\alpha_i}$ and let M_i be the preimage of $\bigoplus_{i=1}^{n} M_{\alpha_i}$.]
   (c) Let S be an uncountable set, k be a field and R be the ring k^S of k-valued functions on S. Let $I \subset R$ be the ideal consisting of functions

f such that $\{s \in S \mid f(s) \neq 0\}$ is at most countable. Show that the R-module I satisfies the second but not the first condition.

(5) In this problem we consider a finite dimensional algebra A over a field k and the Cartan matrix C, $C_{ij} = dim_k(Hom(P_i, P_j))$ where P_i, P_j run over the set of isomorphism classes of indecomposable projective modules.

(a) Let $k = \mathbb{R}$ and assume that A has finite homological dimension. Show that $det(C) = \pm 2^n$ for some n.

(b) (Optional) Let k be an algebraically closed field of characteristic $p > 0$ and $A = k[G]$ for a finite group G which has a normal Sylow p-subgroup. Check [1] that $det(C) = p^n$ for some n.
[Hint. Let $H = G/S$ where S is the Sylow p-subgroup, so that $G = H \ltimes S$ while $k[H]$ is semisimple and irreducible representations of G are in bijection with irreducible representations of H over k and also with irreducible representations of H over a characteristic zero field F. Relate C to the matrix of multiplication by $F[S]$ acting on $K(Rep_F(G))$ and compute this matrix in the basis given by conjugacy classes.]

(6) Let Q be the quiver with two vertices and two edges of opposite orientation connecting them. Set $A = A(Q)/(ab)$ where a, b are the elements corresponding to the two edges.

Show that A is finite dimensional over $\mathbb{C}$ and has finite homological dimension.

[Hint: Recall that an Artinian ring has finite homological dimension provided that each irreducible module has a finite projective resolution.]

[1] In fact, this is true for any finite group G but this general result is harder to prove.

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