# HOMEWORK 4 FOR 18.706, FALL 2018

(1) Let $R \subset Mat_2(\mathbb{Q})$ consist of matrices $A = (a_{ij})$ such that $a_{11} \in \mathbb{Z}$ and $a_{21} = 0$. Show that the homological dimension of $R$ equals two but the homological dimension of $R^{op}$ (i.e. the homological dimension of the category of right $R$-modules) equals one.

(2) Let $D \supset K$ be skew fields. Let $d_l$ be the dimension of $D$ as a left $K$ module and $d_r$ be the dimension of $D$ as a right $K$-module. Show[1] that if $D$ is finite over its center then $d_l = d_r$.

(3) Let $D$ be a skew field which is not algebraic over its center $k$. Show that $R = D \otimes_k k(t)$ is a simple Noetherian ring which is not isomorphic to a matrix ring over a skew field.

[Hint. Define an action of $R$ on $D$. Show that the sum of finitely many copies of that module can not contain a free submodule, deduce $R$ is not a matrix ring over a skew field].

(4) Let $k = \mathbb{F}_p(t_1, t_2)$ be the field of rational functions in two variables over $\mathbb{F}_p$. Set $D = k\langle x, y\rangle/(xy - yx = 1,\ x^p = t_1,\ y^p = t_2)$.

  (a) Show that $D$ is a skew field of dimension $p^2$ over its center $k$. Give an example of a splitting field of $D$.

  (Hint: Check that $\mathbb{F}_p\langle x, y\rangle/(xy - yx = 1)$ and hence $D$ has no zero divisors.)

  (b) (Optional) Let us generalize the definition of $D$ as follows. For $P \in \mathbb{F}_p[t]$ let $D(P)$ be given by $x^p = P(t_1)$, $y^p = t_2$, $xy - yx = 1$.
  Check that $P \to [D(P)]$ is a homomorphism from the additive group of polynomials to $Br(k)$.

  (c) (Optional) Show that the kernel of this homomorphism is $\mathbb{F}_p[t^p]$. Conclude that $p$-torsion in the Brauer group of $k$ is infinite.

(5) Let $A$ be an algebra over a field $k$. Recall that an $n$-th order deformation of $A$ is an associative algebra $\tilde{A}$ over $k[t]/(t^{n+1})$, free as a module over $k[t]/(t^{n+1})$, together with an isomorphism of $k$-algebras $f : \tilde{A}/t\tilde{A} \to A$. Two such deformations $(\tilde{A}, f)$ and $(\tilde{A}', f')$ are said to be equivalent if there exists an algebra isomorphism $g : \tilde{A} \to \tilde{A}'$ such that $f'g = f$. As has been explained in class, first order deformations are parametrized by $HH^2(A)$.

Suppose that $A = Sym(V)$ is a polynomial algebra over a field $k$ of characteristic zero. Recall that Hochschild cohomology of the polynomial algebra is identified with the space of polynomial poly-vector fields on the $V^*$.

Thus a first order deformation of $A = Sym(V)$ is determined by a bivector field $\alpha \in Sym(V) \otimes \wedge^2 V^*$. This bivector field defines a skew-symmetric bilinear operation on $A$, given by $\{f, g\} = \langle \alpha, df \otimes dg\rangle$. Show that the first

[1]According to T.Y. Lam, the question whether $d_l = d_r$ was raised by E. Artin. The answer is negative in general: there exist skew fields $D \supset K$ for which $d_l$, $d_r$ is an arbitrary prescribed pair of integers greater than 1 [Schofield, 1985].

order deformation defined by $\alpha$ lifts to a second order deformation if and only if this operation is a Lie bracket (satisfies the Jacobi identity). In this case $\alpha$ is said to be a *Poisson* bivector field.

(6) Let $A$ be a $k$-algebra for a field $k$, let $H = HH^*(A)$ be its Hochschild cohomology and $H^{ev} = \oplus_n HH^{2n}(A)$ be the even part of $H$. Let $D(A)$ be the category whose objects are $A$-modules and $Hom_{D(A)}(M, N) = Ext^*(M, N)$ with the usual composition maps.

  (a) Define a natural homomorphism $H^{ev} \to End(Id_{D(A)})$.

  (b) Suppose we are given a first order deformation $\tilde{A}$ of $A$ with class $h \in HH^2(A)$ and an $A$-module $M$. Show that the element in $Ext^2_A(M, M)$ assigned to $h$ by part (a) is the obstruction to deforming $M$ to an $\tilde{A}$ module.
  (A deformation of a module $M$ is an $\tilde{A}$-module $\tilde{M}$ free over $k[t]/(t^2)$ and isomorphism $\tilde{M}/t\tilde{M} \cong M$).

  (c) (Optional) Extend the homomorphism in (a) to a homomorphism from $H$ to a modification of $End(Id_{D(A)})$ involving the appropriate sign rule.

(7) (Optional) Let $R$ be the ring of real valued continuous functions on the 2-sphere $S^2$. Let $R^+$ and $R^-$ be the ring of continuous functions on the upper and lower closed hemispheres respectively. Let $A \subset Mat_2(R^+) \times Mat_2(R^-)$ be the subring given by: $m = (m_+, m_-) \in A$ if $m_+(\theta) = S(\theta)m_-(\theta)S(\theta)^{-1}$. Here $\theta \in [0, 2\pi)$ is the standard coordinate on the equator circle bounding the upper and the lower hemisphere, and

$$S(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}.$$

Prove that $A$ is a non-split Azumaya algebra over $R$.

[Hint: Basic topology can be used in this problem. Reduce the statement to the fact that the map $\pi_1(S^1) \to \pi_1(S^1)$ induced by the double cover map $S^1 \to S^1$ is not surjective.]