HOMEWORK 5 FOR 18.706, SPRING 2023

(1) Let F be a field of characteristic different from 2. For $a, b \in F^\times$ let $A_{a,b}$ be the four dimensional algebra over k with basis 1, i, j, k, such that $ij = k = -ji$, $i^2 = a$, $j^2 = b$.
  (a) Check that $A_{a,b}$ is a c.s.a. over F. Let $\left(\frac{a,b}{F}\right)$ denote its class in the Brauer group.
  (b) Show that $\left(\frac{a,b}{F}\right) = 1$ iff $b = Nm_{E/F}(z)$ for some $z \in E = F(\sqrt{a})$. (The operation in the Brauer group is written multiplicatively).
  (c) Check[1] that $\left(\frac{a,b}{F}\right)^2 = 1 = \left(\frac{a,1-a}{F}\right)$ and $\left(\frac{a,bc}{F}\right) = \left(\frac{a,b}{F}\right)\left(\frac{a,c}{F}\right)$

(2) Show that the the ring of integer quaternions $\mathbb{H}_{\mathbb{Z}} = \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{Z}\} \subset \mathbb{H}$ is not an Azumaya algebra over $\mathbb{Z}$, while $\mathbb{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ is an Azumaya algebra over $\mathbb{Z}[\frac{1}{2}]$.

(3) An element x in a ring R is said to be ad locally nilpotent if $ad(x) : a \mapsto xa - ax$ is locally nilpotent, i.e. for any $a \in R$ there exists n such that $ad(x)^n(a) = 0$. Show that a multiplicative set consisting of ad locally nilpotent elements satisfies Ore's condition.

(4) Let K be a skew field and $\phi : K \to K$ a homomorphism. Set $A = K\langle x\rangle/(xa = \phi(a)x)$.

Show that the set of powers of x is a left Ore set, and it is a right Ore set iff ϕ is surjective. In the latter case prove that A is a right Ore domain, i.e. the set of all nonzero elements is right localizing.

(5) Let us say that an ideal $I \subset R$ is right localizable if the set of elements regular modulo I is a right Ore set. (Recall that an element is called regular if it's neither a left nor a right zero divisor).

Let k be a field and $R \subset Mat_2(k[x])$ be given by $R = \{(a_{ij}) \mid a_{21} = 0,\ a_{11} \in k,\ a_{22} - a_{11} \in xk[x]\}$. Show that R is right Noetherian, the ideal of strictly upper triangular matrices is prime, has square zero and is not localizable.

(6) Recall that a module is called uniform if it is nonzero and any two nonzero submodules have a nonzero intersection.

Let R be the ring of continuous $\mathbb{C}$-valued functions on $[0,1]$ with pointwise operations. Show that R has no uniform ideals.

(7) (Optional, repeated from pset 4) Let R be the ring of real valued continuous functions on the 2-sphere S^2. Let R^+ and R^- be the ring of continuous functions on the upper and lower closed hemispheres respectively. Let $A \subset Mat_2(R^+) \times Mat_2(R^-)$ be the subring given by: $m = (m_+, m_-) \in A$ if

[1] The Milnor K_2 group of F is the abelian group generated by symbols $\{a,b\}$, $a,b \in F^\times$ subject to the relations $\{a,bc\} = \{a,b\}\{a,c\}$, $\{a,b\} = \{b,a\}^{-1}$, $\{a,1-a\} = 1$. The identities of this problem yield a homomorphism from $K_2(F)/K_2(F)^2 \to Br(F)[2]$, where $G[2]$ denotes the 2-torsion in an abelian group G. A theorem by Merkuriev (1981) asserts that this homomorphism is an isomorphism.

$m_+(\theta) = S(\theta)m_-(\theta)S(\theta)^{-1}$. Here $\theta \in [0, 2\pi)$ is the standard coordinate on the equator circle bounding the upper and the lower hemisphere, and

$$S(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}.$$

Prove that A is a non-split Azumaya algebra over R.

[Hint: Reduce to the fact that the map $\pi_1(S^1) \to \pi_1(S^1)$ induced by the double cover map $S^1 \to S^1$ is not surjective.]

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