HOMEWORK 6 FOR 18.706, SPRING 2023

(1) Let R be a prime PI-algebra satisfying an identity of degree d. Show that the left (or right) uniform rank of R is less than d.

(2) Prove that if $s \in R$ is regular and ad nilpotent then $GK \dim R[s^{-1}] = GK \dim(R)$.

(3) (GK dimension does not behave well on short exact sequences)

Show that the following provides an example of a PI algebra R, an R-module M with a submodule N, s.t. $GK \dim(N) = GK \dim(M/N) = 1$, $GK \dim(M) = 2$.

Set $R = \mathbb{C}\langle x, y\rangle / yx = 0$, let M have two generators α, β subject to relations: $x^{n+1}y^n\alpha = 0$ and $xy^n\beta = 0$ unless n is a square m^2 in which case $xy^n\beta = xy^m\alpha$. Let $N = R\beta$.

Check that R satisfies the identity $[a, b]^2 = 0$, thus it is PI.

(4) Show that the enveloping algebra $U(sl(2, k))$ is a PI algebra iff the field k has positive characteristic.

(5) Let G be the group of transformations of the line generated by $x \mapsto x + 1$ and $x \mapsto 2x$. Show that the group algebra of G has exponential growth.