18.725: EXERCISE SET 9

DUE THURSDAY NOVEMBER 20

(1) Let $f : X \to Y$ be a morphism of varieties with X complete. Show that there exists a surjective birational morphism $\pi : X' \to X$ from a projective variety X' so that the composite $f' := f \circ \pi$ factors as

$$X' \xrightarrow{\ j\ } Y \times \mathbb{P}^r \xrightarrow{\ p_1\ } Y$$

for some r, where j identifies X' with a closed set of $Y \times \mathbb{P}^r$.

(2) Let $f : X \to Y$ be a morphism of varieties with X complete. Show that $f(X) \subset Y$ is closed and that the variety $f(X)$ is complete.

(3) Assume the fact that any closed analytic sub–manifold $\mathcal{X} \subset \mathbb{P}^n_{\text{an}}$ is induced by a projective variety $X \subset \mathbb{P}^n$ (this fact is difficult and can be found for example in Serre's famous GAGA paper "Géométrie algébrique et géométrie analytique", Ann. Inst. Fourier 6 (1956)). Then show that if X and Y are projective varieties, then any map of analytic spaces $X_{\text{an}} \to Y_{\text{an}}$ is induced by a map of algebraic varieties $X \to Y$.

(4) Let $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ be a hyperplane and let P be a point not in $\mathbb{P}^n$.

(i) For any point $Q \in \mathbb{P}^{n+1} - P$, show that the line through P and Q meets $\mathbb{P}^n$ in exactly one point.

(ii) Define a set map $\pi : \mathbb{P}^{n+1} - P \to \mathbb{P}^n$ by sending Q to the unique point of intersection of the line through P and Q and $\mathbb{P}^n$. Show that π is a morphism.

(5) Let F be the functor on the category of varieties which to any X associates the set $SL_n(\Gamma(X, \mathcal{O}_X))$. That is, n by n matrices with entries in the ring $\Gamma(X, \mathcal{O}_X)$ and determinant one. Show that F is representable by some variety Y. Then explain why there is a distinguished point $e \in Y$ and a morphism

$$m : Y \times Y \longrightarrow Y$$

making Y a group variety (problem set 3). Hint: if you understand Yoneda's lemma this is a very short exercise.

Aside for following problems: Suppose $V \subset \mathbb{P}^n$ is a projective variety with homogeneous coordinate ring

$$R = k[X_0, \dots, X_n]/I(V) = \bigoplus_{d \geq 0} R_d.$$

It is known that there is a polynomial $P(t)$, called the Hilbert polynomial of V, such that for some integer d_0 sufficiently big the dimension of R_d is equal to $P(d)$ for all $d \geq d_0$. It turns out that the dimension of V is equal to the degree of $P(t)$. The *degree* of V is defined to be $(\dim V)!$ times the leading coefficient of $f(t)$.

(6) Calculate the Hilbert polynomial of $V = \mathbb{P}^n$. Verify that $\dim V = n$ and that $\deg V = 1$.

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(7) Suppose $V \subset \mathbb{P}^n$ is equal to $V(f)$ for some $f \in k[X_0, \ldots, X_n]$, where f is irreducible and homogeneous of degree e. Calculate the Hilbert polynomial of V and verify that $\dim V = n-1$ and $\deg V = e$.