## 7. Rational Cherednik algebras and Hecke algebras for varieties with group actions

7.1. **Twisted differential operators.** Let us recall the theory of twisted differential operators (see [BB], section 2).

Let $X$ be a smooth affine algebraic variety over $\mathbb{C}$. Given a closed 2-form $\omega$ on $X$, the algebra $\mathcal{D}_\omega(X)$ of differential operators on $X$ twisted by $\omega$ can be defined as the algebra generated by $\mathcal{O}_X$ and "Lie derivatives" $\mathbf{L}_v$, $v \in \mathrm{Vect}(X)$, with defining relations

$$f\mathbf{L}_v = \mathbf{L}_{fv},\ [\mathbf{L}_v, f] = L_v f,\ [\mathbf{L}_v, \mathbf{L}_w] = \mathbf{L}_{[v,w]} + \omega(v, w).$$

This algebra depends only on the cohomology class $[\omega]$ of $\omega$, and equals the algebra $\mathcal{D}(X)$ of usual differential operators on $X$ if $[\omega] = 0$.

An important special case of twisted differential operators is the algebra of differential operators on a line bundle. Namely, let $L$ be a line bundle on $X$. Since $X$ is affine, $L$ admits an algebraic connection $\nabla$ with curvature $\omega$, which is a closed 2-form on $X$. Then it is easy to show that the algebra $\mathcal{D}(X, L)$ of differential operators on $L$ is isomorphic to $\mathcal{D}_\omega(X)$.

If the variety $X$ is smooth but not necessarily affine, then (sheaves of) algebras of twisted differential operators are classified by the space $\mathrm{H}^2(X, \Omega_X^{\geq 1})$, where $\Omega_X^{\geq 1}$ is the two-step complex of sheaves $\Omega_X^1 \to \Omega_X^{2,\mathrm{cl}}$, given by the De Rham differential acting from 1-forms to closed 2-forms (sitting in degrees 1 and 2, respectively). If $X$ is projective then this space is isomorphic to $\mathrm{H}^{2,0}(X, \mathbb{C}) \oplus \mathrm{H}^{1,1}(X, \mathbb{C})$. We refer the reader to [BB], Section 2, for details.

**Remark 7.1.** One can show that $\mathcal{D}_\omega(X)$ is the universal deformation of $\mathcal{D}(X)$ (see [E1]).

7.2. **Some algebraic geometry preliminaries.** Let $Z$ be a smooth hypersurface in a smooth affine variety $X$. Let $i : Z \to X$ be the corresponding closed embedding. Let $N$ denote the normal bundle of $Z$ in $X$ (a line bundle). Let $\mathcal{O}_X(Z)$ denote the module of regular functions on $X \setminus Z$ which have a pole of at most first order at $Z$. Then we have a natural map of $\mathcal{O}_X$-modules $\phi : \mathcal{O}_X(Z) \to i_*N$. Indeed, we have a natural residue map $\eta : \mathcal{O}_X(Z) \otimes_{\mathcal{O}_X} \Omega_X^1 \to i_*\mathcal{O}_Z$ (where $\Omega_X^1$ is the module of 1-forms), hence a map $\eta' : \mathcal{O}_X(Z) \to i_*\mathcal{O}_Z \otimes_{\mathcal{O}_X} TX = i_*(TX|_Z)$ (where $TX$ is the tangent bundle). The map $\phi$ is obtained by composing $\eta'$ with the natural projection $TX|_Z \to N$.

We have an exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Z) \xrightarrow{\phi} i_*N \to 0$$

Thus we have a natural surjective map of $\mathcal{O}_X$-modules $\xi_Z : TX \to \mathcal{O}_X(Z)/\mathcal{O}_X$.

7.3. **The Cherednik algebra of a variety with a finite group action.** We will now generalize the definition of $H_{t,c}(G, \mathfrak{h})$ to the global case. Let $X$ be an affine algebraic variety over $\mathbb{C}$, and $G$ be a finite group of automorphisms of $X$. Let $E$ be a $G$-invariant subspace of the space of closed 2-forms on $X$, which projects isomorphically to $\mathrm{H}^2(X, \mathbb{C})$. Consider the algebra $G \ltimes \mathcal{O}_{T^*X}$, where $T^*X$ is the cotangent bundle of $X$. We are going to define a deformation $H_{t,c,\omega}(G, X)$ of this algebra parametrized by

(1) complex numbers $t$,
(2) $G$-invariant functions $c$ on the (finite) set $\mathcal{S}$ of pairs $s = (Y, g)$, where $g \in G$, and $Y$ is a connected component of the set of fixed points $X^g$ such that $codim Y = 1$, and
(3) elements $\omega \in E^G = \mathrm{H}^2(X, \mathbb{C})^G$.

If all the parameters are zero, this algebra will conicide with $G \ltimes \mathcal{O}_{T^*X}$.

Let $t, c = \{c(Y,g)\}, \omega \in E^G$ be variables. Let $\mathcal{D}_{\omega/t}(X)_r$ be the algebra (over $\mathbb{C}[t, t^{-1}, \omega]$) of twisted (by $\omega/t$) differential operators on $X$ with rational coefficients.

**Definition 7.2.** A Dunkl-Opdam operator for $(X, G)$ is an element of $\mathcal{D}_{\omega/t}(X)_r[c]$ given by the formula

$$D := t\mathbf{L}_v - \sum_{(Y,g)\in\mathcal{S}} \frac{2c(Y,g)}{1-\lambda_{Y,g}} \cdot f_Y(x) \cdot (1-g), \tag{7.1}$$

where $\lambda_{Y,g}$ is the eigenvalue of $g$ on the conormal bundle to $Y$, $v \in \Gamma(X, TX)$ is a vector field on $X$, and $f_Y \in \mathcal{O}_X(Z)$ is an element of the coset $\xi_Y(v) \in \mathcal{O}_X(Z)/\mathcal{O}_X$ (recall that $\xi_Y$ is defined in Subsection 7.2).

**Definition 7.3.** The algebra $H_{t,c,\omega}(X, G)$ is the subalgebra of $G \ltimes \mathcal{D}_{\omega/t}(X)_r[c]$ generated (over $\mathbb{C}[t, c, \omega]$) by the function algebra $\mathcal{O}_X$, the group $G$, and the Dunkl-Opdam operators.

By specializing $t, c, \omega$ to numerical values, we can define a family of algebras over $\mathbb{C}$, which we will also denote $H_{t,c,\omega}(G, X)$. Note that when we set $t = 0$, the term $t\mathbf{L}_v$ does not become 0 but turns into the classical momentum.

**Definition 7.4.** $H_{t,c,\omega}(G, X)$ is called *the Cherednik algebra of the orbifold $X/G$.*

**Remark 7.5.** One has $H_{1,0,\omega}(G, X) = G \ltimes \mathcal{D}_\omega(X)$. Also, if $\lambda \neq 0$ then $H_{\lambda t,\lambda c,\lambda\omega}(G, X) = H_{t,c,\omega}(G, X)$.

**Example 7.6.** $X = \mathfrak{h}$ is a vector space and $G$ is a subgroup in $\mathrm{GL}(\mathfrak{h})$. Let $v$ be a constant vector field, and let $f_Y(x) = (\alpha_Y, v)/\alpha_Y(x)$, where $\alpha_Y \in \mathfrak{h}^*$ is a nonzero functional vanishing on $Y$. Then the operator $D$ is just the usual Dunkl-Opdam operator $D_v$ in the complex reflection case (see Section 2.5). This implies that all the Dunkl-Opdam operators in the sense of Definition 7.2 have the form $\sum f_i D_{y_i} + a$, where $f_i \in \mathbb{C}[\mathfrak{h}]$, $a \in G \ltimes \mathbb{C}[\mathfrak{h}]$, and $D_{y_i}$ are the usual Dunkl-Opdam operators (for some basis $y_i$ of $\mathfrak{h}$). So the algebra $H_{t,c}(G, \mathfrak{h}) = H_{t,c,0}(G, X)$ is the rational Cherednik algebra for $(G, \mathfrak{h})$, see Section 3.1.

The algebra $H_{t,c,\omega}(G, X)$ has a filtration $F^\bullet$ which is defined on generators by $\deg(\mathcal{O}_X) = \deg(G) = 0$, $\deg(D) = 1$ for Dunkl-Opdam operators $D$.

**Theorem 7.7** (the PBW theorem). *We have*

$$\mathrm{gr}_F(H_{t,c,\omega}(G, X)) = G \ltimes \mathcal{O}(T^*X)[t, c, \omega].$$

*Proof.* Suppose first that $X = \mathfrak{h}$ is a vector space and $G$ is a subgroup in $\mathrm{GL}(\mathfrak{h})$. Then, as we mentioned, $H_{t,c,\omega}(G, \mathfrak{h}) = H_{t,c}(G, \mathfrak{h})$ is the rational Cherednik algebra for $G, \mathfrak{h}$. So in this case the theorem is true.

Now consider arbitrary $X$. We have a homomorphism of graded algebras

$$\psi : \mathrm{gr}_F(H_{t,c,\omega}(G, X)) \to G \ltimes \mathcal{O}(T^*X)[t, c, \omega] \quad \text{(the principal symbol homomorphism).}$$

The homomorphism $\psi$ is clearly surjective, and our job is to show that it is injective (this is the nontrivial part of the proof). In each degree, $\psi$ is a morphism of finitely generated $\mathcal{O}_X^G$-modules. Therefore, to check its injectivity, it suffices to check the injectivity on the formal neighborhood of each point $z \in X/G$.

Let $x$ be a preimage of $z$ in $X$, and $G_x$ be the stabilizer of $x$ in $G$. Then $G_x$ acts on the formal neighborhood $U_x$ of $x$ in $X$.

**Lemma 7.8.** *Any action of a finite group on a formal polydisk over $\mathbb{C}$ is linearizable.*

*Proof.* Let $\mathcal{D}$ be a formal polydisk over $\mathbb{C}$. Suppose we have an action of a finite group $G$ on $\mathcal{D}$. Then we have a group homomorphism:

$$\rho : G \to \mathrm{Aut}(\mathcal{D}) = \mathrm{GL}_n(\mathbb{C}) \ltimes \mathrm{Aut}_U(\mathcal{D}),$$

where $\mathrm{Aut}_U(\mathcal{D})$ is the group of unipotent automorphisms of $\mathcal{D}$ (i.e. those whose derivative at the origin is 1), which is a prounipotent algebraic group.

Our job is to show that the image of $G$ under $\rho$ can be conjugated into $\mathrm{GL}_n(\mathbb{C})$. The obstruction to this is in the cohomology group $\mathrm{H}^1(G, \mathrm{Aut}_U(\mathcal{D}))$, which is trivial since $G$ is finite and $\mathrm{Aut}_U(\mathcal{D})$ is prounipotent over $\mathbb{C}$. $\square$

It follows from Lemma 7.8 that it suffices to prove the theorem in the linear case, which has been accomplished already. We are done. $\square$

**Remark 7.9.** The following remark is meant to clarify the proof of Theorem 7.7. In the case $X = \mathfrak{h}$, the proof of Theorem 7.7 is based, essentially, on the (fairly nontrivial) fact that the usual Dunkl-Opdam operators $D_v$ commute with each other. It is therefore very important to note that in contrast with the linear case, for a general $X$ we do **not** have any natural commuting family of Dunkl-Opdam operators. Instead, the operators (7.1) satisfy a weaker property, which is still sufficient to validate the PBW theorem. This property says that if $D_1, D_2, D_3$ are Dunkl-Opdam operators corresponding to vector fields $v_1, v_2, v_3 := [v_1, v_2]$ and some choices of the functions $f_Y$, then $[D_1, D_2] - D_3 \in G \ltimes \mathcal{O}(X)$ (i.e., it has no poles). To prove this property, it is sufficient to consider the case when $X$ is a formal polydisk, with a linear action of $G$. But in this case everything follows from the commutativity of the usual Dunkl operators $D_v$.

**Example 7.10.** (1) Suppose $G = 1$. Then for $t \neq 0$, $H_{t,0,\omega}(G, X) = \mathcal{D}_{\omega/t}(X)$.
(2) Suppose $G$ is a Weyl group and $X = H$ the corresponding torus. Then $H_{1,c,0}(G, H)$ is called the *trigonometric Cherednik algebra.*

7.4. **Globalization.** Let $X$ be any smooth algebraic variety, and $G \subset \mathrm{Aut}(X)$. Assume that $X$ admits a cover by affine $G$-invariant open sets. Then the quotient variety $X/G$ exists.

For any affine open set $U$ in $X/G$, let $U'$ be the preimage of $U$ in $X$. Then we can define the algebra $H_{t,c,0}(G, U')$ as above. If $U \subset V$, we have an obvious restriction map $H_{t,c,0}(G, V') \to H_{t,c,0}(G, U')$. The gluing axiom is clearly satisfied. Thus the collection of algebras $H_{t,c,0}(G, U')$ can be extended (by sheafification) to a sheaf of algebras on $X/G$. We are going to denote this sheaf by $H_{t,c,0,G,X}$ and call it the sheaf of Cherednik algebras on $X/G$. Thus, $H_{t,c,0,G,X}(U) = H_{t,c,0}(G, U')$.

Similarly, if $\psi \in \mathrm{H}^2(X, \Omega_X^{\geq 1})^G$, we can define the sheaf of twisted Cherednik algebras $H_{t,c,\psi,G,X}$. This is done similarly to the case of twisted differential operators (which is the case $G = 1$).

**Remark 7.11.** (1) The construction of $H_{t,c,\omega}(G, X)$ and the PBW theorem extend in a straightforward manner to the case when the ground field is not $\mathbb{C}$ but an algebraically closed field $k$ of positive characteristic, provided that the order of the group $G$ is relatively prime to the characteristic.

(2) The construction and main properties of the (sheaves of) Cherednik algebras of algebraic varieties can be extended without significant changes to the case when $X$ is a complex analytic manifold, and $G$ is not necessarily finite but acts properly discontinuously. In the following lectures, we will often work in this generalized setting.

7.5. **Modified Cherednik algebra.** It will be convenient for us to use a slight modification of the sheaf $H_{t,c,\psi,G,X}$. Namely, let $\eta$ be a function on the set of conjugacy classes of $Y$ such that $(Y,g)\in\mathcal{S}$. We define $H_{t,c,\eta,\psi,G,X}$ in the same way as $H_{t,c,\psi,G,X}$ except that the Dunkl-Opdam operators are defined by the formula

$$D := t\mathbf{L}_v + \sum_{(Y,g)\in\mathcal{S}} f_Y(x)\frac{2c(Y,g)}{1-\lambda_{Y,g}}(g-1) + \sum_Y f_Y(x)\eta(Y). \tag{7.2}$$

The following result shows that this modification is in fact tautological. Let $\psi_Y$ be the class in $\mathrm{H}^2(X,\Omega_X^{\geq 1})$ defined by the line bundle $\mathcal{O}_X(Y)^{-1}$, whose sections are functions vanishing on $Y$.

**Proposition 7.12.** *One has an isomorphism*

$$H_{t,c,\eta,\psi,G,X} \to H_{t,c,\psi+\sum_Y \eta(Y)\psi_Y,G,X}.$$

*Proof.* Let $y\in Y$ and $z$ be a function on the formal neighborhood of $y$ such that $z|_Y = 0$ and $\mathrm{d}z_y \neq 0$. Extend it to a system of local formal coordinates $z_1 = z, z_2, \ldots, z_d$ near $y$. A Dunkl-Opdam operator near $y$ for the vector field $\frac{\partial}{\partial z}$ can be written in the form

$$D = \frac{\partial}{\partial z} + \frac{1}{z}\Big(\sum_{m=1}^{n-1}\frac{2c(Y,g^m)}{1-\lambda_{Y,g}^m}(g^m-1) + \eta(Y)\Big).$$

Conjugating this operator by the formal expression $z^{\eta(Y)} := (z^m)^{\eta(Y)/m}$, we get

$$z^{\eta(Y)}\circ D\circ z^{-\eta(Y)} = \frac{\partial}{\partial z} + \frac{1}{z}\sum_{m=1}^{n-1}\frac{2c(Y,g^m)}{1-\lambda_{Y,g}^m}(g^m-1)$$

This implies the required statement. ∎

We note that the sheaf $H_{1,c,\eta,0,G,X}$ localizes to $G\ltimes\mathcal{D}_X$ on the complement of all the hypersurfaces $Y$. This follows from the fact that the line bundle $\mathcal{O}_X(Y)$ is trivial on the complement of $Y$.

7.6. **Orbifold Hecke algebras.** Let $X$ be a connected and simply connected complex manifold, and $G$ is a discrete group of automorphisms of $X$ which acts properly discontinuously. Then $X/G$ is a complex orbifold. Let $X'\subset X$ be the set of points with trivial stabilizer. Fix a base point $x_0\in X'$. Then the braid group of $X/G$ is defined to be $B_G = \pi_1(X'/G, x_0)$. We have an exact sequence $1\to K\to B_G\to G\to 1$.

Now let $\mathcal{S}$ be the set of pairs $(Y,g)$ such that $Y$ is a component of $X^g$ of codimension 1 in $X$ (such $Y$ will be called a reflection hypersurface). For $(Y,g)\in\mathcal{S}$, let $G_Y$ be the subgroup of $G$ whose elements act trivially on $Y$. This group is obviously cyclic; let $n_Y = |G_Y|$. Let $C_Y$ be the conjugacy class in $B_G$ corresponding to a small circle going counterclockwise around the image of $Y$ in $X/G$, and $T_Y$ be a representative in $C_Y$.

The following theorem follows from elementary topology:

**Theorem 7.13.** *$K$ is defined by relations $T_Y^{n_Y} = 1$, for all reflection hypersurfaces $Y$ (i.e., $K$ is the intersection of all normal subgroups of $B_G$ containing $T_Y^{n_Y}$).*

*Proof.* See, e.g., [BMR] Proposition 2.17. $\square$

For any conjugacy class of hypersurfaces $Y$ such that $(Y, g) \in \mathcal{S}$ we introduce formal parameters $\tau_{1Y}, \ldots, \tau_{n_Y Y}$. The entire collection of these parameters will be denoted by $\tau$. Let $A_0 = \mathbb{C}[G]$.

**Definition 7.14.** We define the Hecke algebra of $(G, X)$, denoted $A = \mathcal{H}_\tau(G, X, x_0)$, to be the quotient of the group algebra of the braid group, $\mathbb{C}[B_G][[\tau]]$, by the relations

$$\prod_{j=1}^{n_Y} (T - \mathbf{e}^{2\pi j \mathbf{i}/n_Y} \mathbf{e}^{\tau_{jY}}) = 0, \; T \in C_Y \tag{7.3}$$

(i.e., by the closed ideal in the formal series topology generated by these relations).

Thus, $A$ is a deformation of $A_0$.

It is clear that up to an isomorphism this algebra is independent on the choice of $x_0$, so we will sometimes drop $x_0$ form the notation.

The main result of this section is the following theorem.

**Theorem 7.15.** *Assume that $\mathrm{H}^2(X, \mathbb{C}) = 0$. Then $A = \mathcal{H}_\tau(G, X)$ is a* **flat** *formal deformation of $A_0$, which means $A = A_0[[\tau]]$ as a module over $\mathbb{C}[[\tau]]$.*

**Example 7.16.** Let $\mathfrak{h}$ be a finite dimensional vector space, and $G$ be a complex reflection group in $\mathrm{GL}(\mathfrak{h})$. Then $\mathcal{H}_\tau(G, \mathfrak{h})$ is the Hecke algebra of $G$ studied in [BMR]. It follows from Theorem 7.15 that this Hecke algebra is flat. This proof of flatness is in fact the same as the original proof of this result given in [BMR] (based on the Dunkl-Opdam-Cherednik operators, and explained above).

**Example 7.17.** Let $\mathfrak{h}$ be a universal covering of a maximal torus of a simply connected simple Lie group $G$, $Q^\vee$ be the dual root lattice, and $\widehat{G} = G \ltimes Q^\vee$ be its affine Weyl group. Then $\mathcal{H}_\tau(\mathfrak{h}, \widehat{G})$ is the affine Hecke algebra. This algebra is also flat by Theorem 7.15. In fact, its flatness is a well known result from representation theory; our proof of flatness is essentially due to Cherednik [Ch].

**Example 7.18.** Let $G, \mathfrak{h}, Q^\vee$ be as in the previous example, $\eta \in \mathbb{C}_+$ be a complex number with a positive imaginary part, and $\widehat{\widehat{G}} = G \ltimes (Q^\vee \oplus \eta Q^\vee)$ be the double affine Weyl group. Then $\mathcal{H}_\tau(\mathfrak{h}, \widehat{\widehat{G}})$ is (one of the versions of) the double affine Hecke algebra of Cherednik ([Ch]), and it is flat by Theorem 7.15. The fact that this algebra is flat was proved by Cherednik, Sahi, Noumi, Stokman (see [Ch],[Sa],[NoSt],[St]) using a different approach (q-deformed Dunkl operators).

7.7. **Hecke algebras attached to Fuchsian groups.** Let $H$ be a simply connected complex Riemann surface (i.e., Riemann sphere, Euclidean plane, or Lobachevsky plane), and $\Gamma$ be a cocompact lattice in $\mathrm{A}ut(H)$ (i.e., a Fuchsian group). Let $\Sigma = H/\Gamma$. Then $\Sigma$ is a compact complex Riemann surface. When $\Gamma$ contains elliptic elements (i.e., nontrivial elements of finite order), we are going to regard $\Sigma$ as an orbifold: it has special points $P_i$, $i = 1, \dots, m$ with stabilizers $\mathbb{Z}_{n_i}$. Then $\Gamma$ is the orbifold fundamental group of $\Sigma$.[1]

Let $g$ be the genus of $\Sigma$, and $a_l, b_l, l = 1, \dots, g$, be the $a$-cycles and $b$-cycles of $\Sigma$. Let $c_j$ be the counterclockwise loops around $P_j$. Then $\Gamma$ is generated by $a_l, b_l, c_j$ with relations

$$c_j^{n_j} = 1,\ c_1 c_2 \cdots c_m = \prod_l a_l b_l a_l^{-1} b_l^{-1}.$$

For each $j$, introduce formal parameters $\tau_{kj}$, $k = 1, \dots, n_j$. Define the Hecke algebra $\mathcal{H}_\tau(\Sigma)$ of $\Sigma$ to be generated over $\mathbb{C}[[\tau]]$ by the same generators $a_l, b_l, c_j$ with defining relations

$$\prod_{k=1}^{n_j} (c_j - \mathbf{e}^{2\pi j \mathbf{i}/n_j} \mathbf{e}^{\tau_{kj}}) = 0,\ c_1 c_2 \cdots c_m = \prod_l a_l b_l a_l^{-1} b_l^{-1}.$$

Thus $\mathcal{H}_\tau(\Sigma)$ is a deformation of $\mathbb{C}[\Gamma]$.

This deformation is flat if $H$ is a Euclidean plane or a Lobachevsky plane. Indeed, $\mathcal{H}_\tau(\Sigma) = \mathcal{H}_\tau(\Gamma, H)$, so the result follows from Theorem 7.15 and the fact that $\mathrm{H}^2(H, \mathbb{C}) = 0$.

On the other hand, if $H$ is the Riemann sphere (so that the condition $\mathrm{H}^2(H, \mathbb{C}) = 0$ is violated) and $\Gamma \neq 1$ then this deformation is not flat. Indeed, let $\tau = \tau(\hbar)$ be a 1-parameter subdeformation of $\mathcal{H}_\tau(\Sigma)$ which is flat. Let us compute the determinant of the product $c_1 \cdots c_m$ in the regular representation of this algebra (which is finite dimensional if $H$ is the sphere). On the one hand, it is 1, as $c_1 \cdots c_m$ is a product of commutators. On the other hand, the eigenvalues of $c_j$ in this representation are $\mathbf{e}^{2\pi j \mathbf{i}/n_j} \mathbf{e}^{\tau_{kj}}$ with multiplicity $|\Gamma|/n_j$. Computing determinants as products of eigenvalues, we get a nontrivial equation on $\tau_{kj}(\hbar)$, which means that the deformation $\mathcal{H}_\tau$ is not flat.

Thus, we see that $\mathcal{H}_\tau(\Sigma)$ fails to be flat in the following "forbidden" cases:

$$g = 0,\ m = 2, (n_1, n_2) = (n, n);$$

$$m = 3, (n_1, n_2, n_3) = (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5).$$

Indeed, the orbifold Euler characteristic of a closed surface $\Sigma$ of genus $g$ with $m$ special points $x_1, \dots, x_m$ whose orders are $n_1, \dots, n_m$ is

$$\chi^{\mathrm{orb}}(\Sigma, x_1, \dots, x_m) = 2 - 2g - m + \sum_{i=1}^m \frac{1}{n_i},$$

and above solutions are the solutions of the inequality

$$\chi^{\mathrm{orb}}(\mathbb{C}P^1, x_1, \dots, x_m) > 0.$$

(note that the solutions for $m = 1$ and solutions $(n_1, n_2)$ with $n_1 \neq n_2$ don't arise, since they don't correspond to any orbifolds).

[1]Let $X$ be a connected topological space on with a properly discontinuous action of a discrete group $G$. Then the orbifold fundamental group of the orbifold $X/G$ with base point $x \in X$, denoted $\pi_1^{orb}(X/G, x)$, is the set of pairs $(g, \gamma)$, where $g \in G$ and $\gamma$ is a homotopy class of paths leading from $x$ to $gx$, with multiplication law $(g_1, \gamma_1)(g_2, \gamma_2) = (g_1 g_2, \gamma)$, where $\gamma$ is $\gamma_1$ followed by $g_1(\gamma_2)$. Obviously, in this situation we have an exact sequence

$$1 \to \pi_1(X, x) \to \pi_1^{orb}(X/G, x) \to G \to 1.$$

7.8. **Hecke algebras of wallpaper groups and del Pezzo surfaces.** The case when $H$ is the Euclidean plane (i.e., $\Gamma$ is a wallpaper group) deserves special attention. If there are elliptic elements, this reduces to the following configurations: $g = 0$ and

$$m = 3, (n_1, n_2, n_3) = (3, 3, 3), (2, 4, 4), (2, 3, 6) \text{ (cases } E_6,\ E_7,\ E_8),$$

or

$$m = 4, (n_1, n_2, n_3, n_4) = (2, 2, 2, 2) \text{ (case } D_4).$$

In these cases, the algebra $\mathcal{H}_\tau(\Gamma, H)$ (for numerical $\tau$) has Gelfand-Kirillov dimension 2, so it can be interpreted in terms of the theory of noncommutative surfaces.

Recall that a del Pezzo surface (or a Fano surface) is a smooth projective surface, whose anticanonical line bundle is ample. It is known that such surfaces are $\mathbb{CP}^1 \times \mathbb{CP}^1$, or a blow-up of $\mathbb{CP}^2$ at up to 8 generic points. The degree of a del Pezzo surface $X$ is by definition the self intersection number $K \cdot K$ of its canonical class $K$. For example, a del Pezzo surface of degree 3 is a cubic surface in $\mathbb{CP}^3$, and the degree of $\mathbb{CP}^2$ with $n$ generic points blown up is $d = 9 - n$.

Now suppose $\tau$ is numerical. Let $\hbar = \sum_{j,k} n_j^{-1} \tau_{kj}$. Also let $n$ be the largest of $n_j$, and $c$ be the corresponding $c_j$. Let $\mathsf{e} \in \mathbb{C}[c] \subset \mathcal{H}_\tau(\Gamma, H)$ be the projector to an eigenspace of $c$. Consider the "spherical" subalgebra $B_\tau(\Gamma, H) := \mathsf{e}\mathcal{H}_\tau(\Gamma, H)\mathsf{e}$.

**Theorem 7.19** (Etingof, Oblomkov, Rains, [EOR])**.** (i) *If $\hbar = 0$ then the algebra $B_\tau(\Gamma, H)$ is commutative, and its spectrum is an affine del Pezzo surface. More precisely, in the case $(2, 2, 2, 2)$, it is a del Pezzo surface of degree 3 (a cubic surface) with a triangle of lines removed; in the cases $(3, 3, 3), (2, 4, 4), (2, 3, 6)$ it is a del Pezzo surface of degrees 3,2,1 respectively with a nodal rational curve removed.*

(ii) *The algebra $B_\tau(\Gamma, H)$ for $\hbar \neq 0$ is a quantization of the unique algebraic symplectic structure on the surface from (i) with Planck's constant $\hbar$.*

*Proof.* See [EOR]. □

**Remark 7.20.** In the case $(2, 2, 2, 2)$, $\mathcal{H}_\tau(\Gamma, \Gamma)$ is the Cherednik-Sahi algebra of rank 1; it controls the theory of Askey-Wilson polynomials.

**Example 7.21.** This is a "multivariate" version of the Hecke algebras attached to Fuchsian groups, defined in the previous subsection. Namely, letting $\Gamma, H$ be as in the previous subsection, and $N \geq 1$, we consider the manifold $X = H^N$ with the action of $\Gamma_N = \mathfrak{S}_N \ltimes \Gamma^N$. If $H$ is a Euclidean or Lobachevsky plane, then by Theorem 7.15 $\mathcal{H}_\tau(\Gamma_N, X^N)$ is a flat deformation of the group algebra $\mathbb{C}[\Gamma_N]$. If $N > 1$, this algebra has one more essential parameter than for $N = 1$ (corresponding to reflections in $\mathfrak{S}_N$). In the Euclidean case, one expects that an appropriate "spherical" subalgebra of this algebra is a quantization of the Hilbert scheme of a del Pezzo surface.

7.9. **The Knizhnik-Zamolodchikov functor.** In this subsection we will define a global analog of the KZ functor defined in [GGOR]. This functor will be used as a tool of proof of Theorem 7.15.

Let $X$ be a simply connected complex manifold, and $G$ a discrete group of holomorphic transformations of $X$ acting on $X$ properly discontinuously. Let $X' \subset X$ be the set of points with trivial stabilizer. Then we can define the sheaf of Cherednik algebras $H_{1,c,\eta,0,G,X}$ on $X/G$. Note that the restriction of this sheaf to $X'/G$ is the same as the restriction of the sheaf $G \ltimes \mathcal{D}_X$ to $X'/G$ (i.e. on $X'/G$, the dependence of the sheaf on the parameters $c$ and $\eta$ disappears). This follows from the fact that the line bundles $\mathcal{O}_X(Y)$ become trivial when restricted to $X'$.

Now let $M$ be a module over $H_{1,c,\eta,0,G,X}$ which is a locally free coherent sheaf when restricted to $X'/G$. Then the restriction of $M$ to $X'/G$ is a $G$-equivariant $\mathcal{D}$-module on $X'$ which is coherent and locally free as an $\mathcal{O}$-module. Thus, $M$ corresponds to a locally constant sheaf (local system) on $X'/G$, which gives rise to a monodromy representation of the braid group $\pi_1(X'/G, x_0)$ (where $x_0$ is a base point). This representation will be denoted by $\mathrm{KZ}(M)$. This defines a functor KZ, which is analogous to the one in [GGOR].

It follows from the theory of $\mathcal{D}$-modules that any $\mathcal{O}_{X/G}$-coherent $H_{1,c,\eta,0,G,X}$-module is locally free when restricted to $X'/G$. Thus the KZ functor acts from the abelian category $\mathcal{C}_{c,\eta}$ of $\mathcal{O}_{X/G}$-coherent $H_{1,c,\eta,0,G,X}$-modules to the category of representations of $\pi_1(X'/G, x_0)$. It is easy to see that this functor is exact.

For any $Y$, let $g_Y$ be the generator of $G_Y$ which has eigenvalue $\mathbf{e}^{2\pi \mathbf{i}/n_Y}$ in the conormal bundle to $Y$. Let $(c,\eta) \to \tau(c,\eta)$ be the invertible linear transformation defined by the formula

$$\tau_{jY} = 2\pi\mathbf{i}(2\sum_{m=1}^{n_Y-1} c(Y, g_Y^m)\frac{1-\mathbf{e}^{2\pi jm\mathbf{i}/n_Y}}{1-\mathbf{e}^{-2\pi m\mathbf{i}/n_Y}} - \eta(Y))/n_Y.$$

**Proposition 7.22.** *The functor* KZ *maps the category $\mathcal{C}_{c,\eta}$ to the category of representations of the algebra $\mathcal{H}_{\tau(c,\eta)}(G,X)$.*

*Proof.* The result follows from the corresponding result in the linear case (which we have already proved) by restricting $M$ to the union of $G$-translates of a neighborhood of a generic point $y \in Y$, and then linearizing the action of $G_Y$ on this neighborhood. □

7.10. **Proof of Theorem 7.15.** Consider the module $M = \mathrm{Ind}_{\mathcal{D}_X}^{G\ltimes\mathcal{D}_X}\mathcal{O}_X$. Then $\mathrm{KZ}(M)$ is the regular representation of $G$ which is denoted by $\mathrm{reg}G$. We want to show that $M$ deforms uniquely (up to an isomorphism) to a module over $H_{1,c,0,\eta,G,X}$ for formal $c,\eta$. The obstruction to this deformation is in $\mathrm{Ext}^2_{G\ltimes\mathcal{D}_X}(M,M)$ and the freedom of this deformation is in $\mathrm{Ext}^1_{G\ltimes\mathcal{D}_X}(M,M)$. Since

$$\mathrm{Ext}^i_{G\ltimes\mathcal{D}_X}(M,M) = \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{O}_X, \mathrm{Res}M) = \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X\otimes\mathbb{C}G)$$

$$= \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{O}_X)\otimes\mathbb{C}G = \mathrm{H}^i(X,\mathbb{C})\otimes\mathbb{C}G,$$

and $X$ is simply connected, we have

$$\mathrm{Ext}^1_{G\ltimes\mathcal{D}_X}(M,M) = 0, \text{ and } \mathrm{Ext}^2_{G\ltimes\mathcal{D}_X}(M,M) = 0 \text{ if } \mathrm{H}^2(X,\mathbb{C}) = 0.$$

Thus such deformation exists and is unique if $\mathrm{H}^2(X,\mathbb{C}) = 0$.

Now let $M_{c,\eta}$ be the deformation. Then $\mathrm{KZ}(M_{c,\eta})$ is a $\mathcal{H}_{\tau(c,\eta)}(G,X)$-module from Proposition 7.22 and it deforms flatly the module $\mathrm{reg}G$. This implies $\mathcal{H}_{\tau(c,\eta)}(G,X)$ is flat over $\mathbb{C}[[\tau]]$.

**Remark 7.23.** When $X$ is not simply connected, the theorem is still true under the assumption $\pi_2(X)\otimes\mathbb{C} = 0$ (i.e. $\mathrm{H}^2(\widetilde{X},\mathbb{C}) = 0$, where $\widetilde{X}$ is the universal cover of $X$), and the proof is contained in [E1].

7.11. **Example: the simplest case of double affine Hecke algebras.** Now let $G = \mathbb{Z}_2 \ltimes \mathbb{Z}^2$ acting on $\mathbb{C}$. Then the conjugacy classes of reflection hyperplanes are four points: $0, 1/2, 1/2+\eta/2, \eta/2$, where we suppose the lattice in $\mathbb{C}$ is $\mathbb{Z} \oplus \mathbb{Z}\eta$. Correspondingly, the presentation of $G$ is as follows:

$$\textbf{generators: } T_1, T_2, T_3, T_4; \qquad \textbf{relations: } T_1T_2T_3T_4 = 1, T_i^2 = 1.$$

Thus, the corresponding orbifold Hecke algebra is the following deformation of $\mathbb{C}G$:

$$\textbf{generators: } T_1, T_2, T_3, T_4; \qquad \textbf{relations: } T_1T_2T_3T_4 = 1, (T_i - p_i)(T_i - q_i) = 0,$$

where $p_i, q_i$ $(i = 1, \dots, 4)$, are parameters.

If we renormalize the $T_i$, these relations turn into

$$(T_i - t_i)(T_i + t_i^{-1}) = 0, \quad T_1T_2T_3T_4 = q,$$

and we get the type $C^\vee C_1$ double affine Hecke algebra. If we set three of the four $T_i$'s satisfying the undeformed relation $T_i^2 = 1$, we get the double affine Hecke algebra of type $A_1$. More precisely, this algebra is generated by $T_1, \dots, T_4$ with relations

$$T_2^2 = T_3^2 = T_4^2 = 1, \quad (T_1 - t)(T_1 + t^{-1}) = 0, \quad T_1T_2T_3T_4 = q.$$

Another presentation of this algebra (which is more widely used) is as follows. Let $E = \mathbb{C}/\mathbb{Z}^2$, an elliptic curve with an $\mathbb{Z}_2$ action defined by $z \mapsto -z$. Define *the partial braid group*

$$B = \pi_1^{\mathrm{orb}}(E\backslash\{0\}/\mathbb{Z}_2, x),$$

where $x$ is a generic point. Notice that comparing to the usual braid group, we do not delete three of the four reflection points. The generators of the group $\pi_1(E \setminus \{0\}, x)$ (the fundamental group of a punctured 2-torus) are $X$ (corresponding to the $a$-cycle on the torus), $Y$ (corresponding to the $b$-cycle on the torus) and $C$ (corresponding to the loop around 0). In order to construct $B$, which is an extension of $\mathbb{Z}_2$ by $\pi_1(E \setminus \{0\}, x)$, we introduce an element $T$ s.t. $T^2 = C$ (the half-loop around the puncture). Then $X, Y, T$ satisfy the following relations:

$$TXT = X^{-1}, \quad T^{-1}YT^{-1} = Y^{-1}, \quad Y^{-1}X^{-1}YXT^2 = 1.$$

The Hecke algebra of the partial braid group is then defined to be the group algebra of $B$ plus an extra relation: $(T - q_1)(T + q_2) = 0$.

A common way to present this Hecke algebra is to renormalize the generators so that one has the following relations:

$$TXT = X^{-1}, T^{-1}YT^{-1} = Y^{-1}, Y^{-1}X^{-1}YXT^2 = q, (T - t)(T + t^{-1}) = 0.$$

This is Cherednik's definition for $\mathcal{HH}(q, t)$, the double affine Hecke algebra of type $A_1$ which depends on two parameters $q, t$.

There are two degenerations of the algebra $\mathcal{HH}(q, t)$.

1. **The trigonometric degeneration.**

Set $Y = \mathbf{e}^{\hbar y}$, $q = \mathbf{e}^{\hbar}$, $t = \mathbf{e}^{\hbar c}$ and $T = s\mathbf{e}^{\hbar cs}$, where $s \in \mathbb{Z}_2$ is the reflection. Then $s, X, y$ satisfy the following relations modulo $\hbar$:

$$s^2 = 1, \quad sXs^{-1} = X^{-1}, \quad sy + ys = 2c, \quad X^{-1}yX - y = 1 - 2cs.$$

The algebra generated by $s, X, y$ with these relations is called the type $A_1$ trigonometric Cherednik algebra. It is easy to show that it is isomorphic to the Cherednik algebra $H_{1,c}(\mathbb{Z}_2, \mathbb{C}^*)$, where $\mathbb{Z}_2$ acts on $\mathbb{C}^*$ by $z \to z^{-1}$.

2. **The rational degeneration.**

In the trigonometric Cherednik algebra, set $X = \mathbf{e}^{\hbar x}$ and $y = \hat{y}/\hbar$. Then $s, x, \hat{y}$ satisfy the following relations modulo $\hbar$:

$$s^2 = 1, sx = -xs, s\hat{y} = -\hat{y}s, \hat{y}x - x\hat{y} = 1 - 2cs.$$

The algebra generated by $s, x, \hat{y}$ with these relations is the rational Cherednik algebra $H_{1,c}(\mathbb{Z}_2, \mathbb{C})$ with the action of $\mathbb{Z}_2$ on $\mathbb{C}$ is given by $z \to -z$.

7.12. **Affine and extended affine Weyl groups.** Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system with respect to a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on $\mathbb{R}^n$. We will assume that $R$ is reduced. Let $\{\alpha_i\}_{i=1}^n \subset R$ be the set of simple roots and $R_+$ (respectively $R_-$) be the set of positive (respectively negative) roots. The coroots are denoted by $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. Let $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ be the coroot lattice and $P^\vee = \bigoplus_{i=1}^n \mathbb{Z}\omega_i^\vee$ the coweight lattice, where $\omega_i^\vee$'s are the fundamental coweights, i.e., $(\omega_i^\vee, \alpha_j) = \delta_{ij}$. Let $\theta$ be the maximal positive root, and assume that the bilinear form is normalized by the condition $(\theta,\theta) = 2$. Let $\overline{W}$ be the Weyl group which is generated by the reflections $s_\alpha$ ($\alpha \in R$).

By definition, the affine root system is

$$R^a = \{\tilde{\alpha} = [\alpha, j] \in \mathbb{R}^n \times \mathbb{R} | \text{ where } \alpha \in R, j \in \mathbb{Z}\}.$$

The set of positive affine roots is $R_+^a = \{[\alpha, j] \,|\, j \in \mathbb{Z}_{>0}\} \cup \{[\alpha, 0] \,|\, \alpha \in R_+\}$. Define $\alpha_0 = [-\theta, 1]$. We will identify $\alpha \in R$ with $\tilde{\alpha} = [\alpha, 0] \in R^a$.

For an arbitrary affine root $\tilde{\alpha} = [\alpha, j]$ and a vector $\tilde{z} = [z, \zeta] \in \mathbb{R}^n \times \mathbb{R}$, the corresponding affine reflection is defined as follows:

$$s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - 2\frac{(z,\alpha)}{(\alpha,\alpha)}\,\tilde{\alpha} = \tilde{z} - (z, \alpha^\vee)\,\tilde{\alpha}.$$

The affine Weyl group $\overline{W}_a$ is generated by the affine reflections $\{s_{\tilde{\alpha}} \,|\, \tilde{\alpha} \in \widetilde{R}_+\}$, and we have an isomorphism:

$$\overline{W}_a \cong \overline{W} \ltimes Q^\vee,$$

where the translation $\alpha^\vee \in Q^\vee$ is naturally identified with the composition $s_{[-\alpha,1]}s_\alpha \in \overline{W}_a$.

Define the extended affine Weyl group to be $\overline{W}_a^{\text{ext}} = \overline{W} \ltimes P^\vee$ acting on $\mathbb{R}^{n+1}$ via $b(\tilde{z}) = [z, \zeta - (b, z)]$ for $\tilde{z} = [z, \zeta]$, $b \in P^\vee$. Then $\overline{W}_a \subset \overline{W}_a^{\text{ext}}$. Moreover, $\overline{W}_a$ is a normal subgroup of $\overline{W}_a^{\text{ext}}$ and $\overline{W}_a^{\text{ext}}/\overline{W}_a = P^\vee/Q^\vee$. The latter group can be identified with the group $\Pi = \{\pi_r\}$ of the elements of $\overline{W}_a^{\text{ext}}$ permuting simple affine roots under their action in $\mathbb{R}^{n+1}$. It is a normal commutative subgroup of $\mathrm{Aut} = \mathrm{Aut}(\mathrm{Dyn}^a)$ ($\mathrm{Dyn}^a$ denotes the affine Dynkin diagram). The quotient $\mathrm{Aut}/\Pi$ is isomorphic to the group of the automorphisms preserving $\alpha_0$, i.e. the group $\mathrm{AutDyn}$ of automorphisms of the finite Dynkin diagram.

7.13. **Cherednik's double affine Hecke algebra of a root system.** In this subsection, we will give an explicit presentation of Cherednik's DAHA for a root system, defined in Example 7.18. This is done by giving an explicit presentation of the corresponding braid group (which is called *the elliptic braid group*), and then imposing quadratic relations on the generators corresponding to reflections.

For a root system $R$, let $m = 2$ if $R$ is of type $D_{2k}$, $m = 1$ if $R$ is of type $B_{2k}, C_k$, and otherwise $m = |\Pi|$. Let $m_{ij}$ be the number of edges between vertex $i$ and vertex $j$ in the affine Dynkin diagram of $R^a$. Let $X_i$ $(i=1,\ldots,n)$ be a family of pairwise commutative and algebraically independent elements. Set

$$X_{[b,j]} = \prod_{i=1}^{n} X_i^{\ell_i} q^j, \text{ where } b = \sum_{i=1}^{n} \ell_i \omega_i \in P, j \in \mathbb{Z}/m\mathbb{Z}.$$

For an element $\hat{w} \in \overline{W}_a^{\text{ext}}$, we can define an action on these $X_{[b,j]}$ by $\hat{w} X_{[b,j]} = X_{\hat{w}[b,j]}$.

**Definition 7.24** (Cherednik)**.** *The double affine Hecke algebra* (DAHA) of the root system $R$, denoted by $\mathcal{HH}$, is an algebra defined over the field $\mathbb{C}_{q,t} = \mathbb{C}(q^{1/m}, t^{1/2})$, generated by $T_i, i = 0,\ldots,n, \Pi, X_b, b \in P$, subject to the following relations:

(1) $T_iT_jT_i \cdots = T_jT_iT_j \cdots$, $m_{ij}$ factors each side;
(2) $(T_i - t_i)(T_i + t_i^{-1}) = 0$ for $i = 0,\ldots,n$;
(3) $\pi T_i \pi^{-1} = T_{\pi(i)}$, for $\pi \in \Pi$ and $i = 0,\ldots,n$;
(4) $\pi X_b \pi^{-1} = X_{\pi(b)}$, for $\pi \in \Pi, b \in P$;
(5) $T_i X_b T_i = X_b X_{\alpha_i}^{-1}$, if $i > 0$ and $(b, \alpha_i^\vee) = 1$; $T_i X_b = X_b T_i$, if $i > 0$ and $(b, \alpha_i^\vee) = 0$;
(6) $T_0 X_b T_0 = X_{b-\alpha_0}$ if $(b,\theta) = -1$; $T_0 X_b = X_b T_0$ if $(b,\theta) = 0$.

Here $t_i$ are parameters attached to simple affine roots (so that roots of the same length give rise to the same parameters).

*The degenerate double affine Hecke algebra (trigonometric Cherednik algebra) $\mathcal{HH}_{trig}$* is generated by the group algebra of $\overline{W}_a^{\text{ext}}$, $\Pi$ and pairwise commutative $y_{\tilde{b}} = \sum_{i=1}^n (b, \alpha^\vee) y_i + u$ for $\tilde{b} = [b,u] \in P \times \mathbb{Z}$, with the following relations:

$$s_i y_b - y_{s_i(b)} s_i = -k_i (b, \alpha_i^\vee), \text{ for } i = 1,\ldots,n,$$
$$s_0 y_b - y_{s_0(b)} s_0 = k_0(b,\theta), \qquad \pi_r y_b \pi_r^{-1} = y_{\pi_r(b)} \text{ for } \pi_r \in \Pi.$$

**Remark 7.25.** This degeneration can be obtained from the DAHA similarly to the case of $A_1$, which is described above.

7.14. **Algebraic flatness of Hecke algebras of polygonal Fuchsian groups.** Let $W$ be the Coxeter group of rank $r$ corresponding to a Coxeter datum:

$$m_{ij}\ (i,j = 1,\ldots,r, i \neq j), \text{ such that } 2 \leq m_{ij} \leq \infty \text{ and } m_{ij} = m_{ji}.$$

So the group $W$ has generators $s_i$ $i = 1,\ldots,r$, and defining relations

$$s_i^2 = 1, \ (s_i s_j)^{m_{ij}} = 1 \text{ if } m_{ij} \neq \infty.$$

It has a sign character $\xi : W \to \{\pm 1\}$ given by $\xi(s_i) = -1$. Denote by $W_+$ the kernel of $\xi$ (the even subgroup of $W$). It is generated by $a_{ij} = s_i s_j$ with relations:

$$a_{ij} = a_{ji}^{-1}, \quad a_{ij} a_{jk} a_{ki} = 1, \quad a_{ij}^{m_{ij}} = 1.$$

We can deform the group algebra $\mathbb{C}[W]$ as follows. Define the algebra $A(W)$ with invertible generators $s_i$, and $t_{ij,k}$, $i,j = 1,\ldots,r$, $k \in \mathbb{Z}_{m_{ij}}$ for $(i,j)$ such that $m_{ij} < \infty$ and defining relations

$$t_{ij,k} = t_{ji,-k}^{-1}, \quad s_i^2 = 1, \quad [t_{ij,k}, t_{i'j',k'}] = 0, \quad s_p t_{ij,k} = t_{ji,k} s_p,$$
$$\prod_{k=1}^{m_{ij}} (s_i s_j - t_{ij,k}) = 0 \text{ if } m_{ij} < \infty.$$

Notice that if we set $t_{ij,k} = \exp(2\pi k \mathbf{i}/m_{ij})$, we get $\mathbb{C}[W]$.

Define also the algebra $A_+(W)$ over $\mathcal{R} := \mathbb{C}[t_{ij,k}]$ $(t_{ij,k} = t_{ji,-k}^{-1})$ by generators $a_{ij}$, $i \neq j$ $(a_{ij} = a_{ji}^{-1})$, and relations

$$\prod_{k=1}^{m_{ij}} (a_{ij} - t_{ij,k}) = 0 \text{ if } m_{ij} < \infty, \quad a_{ij}a_{jp}a_{pi} = 1.$$

If $w$ is a word in letters $s_i$, let $T_w$ be the corresponding element of $A(W)$. Choose a function $w(x)$ which attaches to every element $x \in W$, a reduced word $w(x)$ representing $x$ in $W$.

**Theorem 7.26** (Etingof, Rains, [ER]). (i) *The elements $T_{w(x)}$, $x \in W$, form a spanning set in $A(W)$ as a left $\mathcal{R}$-module.*

(ii) *The elements $T_{w(x)}$, $x \in W_+$, form a spanning set in $A_+(W)$ as a left $\mathcal{R}$-module.*

(iii) *The elements $T_{w(x)}$, $x \in W$, are linearly independent if $W$ has no finite parabolic subgroups of rank* 3.

*Proof.* We only give the proof of (i). Statement (ii) follows from (i). Proof of (iii), which is quite nontrivial, can be found in [ER] (it uses the geometry of constructible sheaves on the Coxeter complex of $W$).

Let us write the relation

$$\prod_{k=1}^{m_{ij}} (s_i s_j - t_{ij,k}) = 0$$

as a deformed braid relation:

$$s_j s_i s_j \ldots + S.L.T. = t_{ij} s_i s_j s_i \ldots + S.L.T.,$$

where $t_{ij} = (-1)^{m_{ij}+1} t_{ij,1} \cdots t_{ij,m_{ij}}$, S.L.T. mean "smaller length terms", and the products on both sides have length $m_{ij}$. This can be done by multiplying the relation by $s_i s_j \cdots$ ($m_{ij}$ factors).

Now let us show that $T_{w(x)}$ span $A(W)$ over $\mathcal{R}$. Clearly, $T_w$ for all words $w$ span $A(W)$. So we just need to take any word $w$ and express $T_w$ via $T_{w(x)}$.

It is well known from the theory of Coxeter groups (see e.g. [B]) that using the braid relations, one can turn any non-reduced word into a word that is not square free, and any reduced expression of a given element of $W$ into any other reduced expression of the same element. Thus, if $w$ is non-reduced, then by using the deformed braid relations we can reduce $T_w$ to a linear combination of $T_u$ with words $u$ of smaller length than $w$. On the other hand, if $w$ is a reduced expression for some element $x \in W$, then using the deformed braid relations we can reduce $T_w$ to a linear combination of $T_u$ with $u$ shorter than $w$, and $T_{w(x)}$. Thus $T_{w(x)}$ are a spanning set. This proves (i). □

Thus, $A_+(W)$ is a "deformation" of $\mathbb{C}[W_+]$ over $\mathcal{R}$, and similarly $A(W)$ is a "twisted deformation" of $\mathbb{C}[W]$.

Now let $\Gamma = \Gamma(m_1, \ldots, m_r)$, $r \geq 3$, be the Fuchsian group defined by generators $c_j$, $j = 1, \ldots, r$, with defining relations

$$c_j^{m_j} = 1, \; \prod_{j=1}^{r} c_j = 1.$$

Here $2 \leq m_j < \infty$.

Suppose $\Gamma$ acts on $H$ where $H$ is a simply connected complex Riemann surface as in Section 7.7. We have the Hecke algebra of $\Gamma$, $\mathcal{H}_\tau(H,\Gamma)$, defined by the same (invertible) generators $c_j$ and relations

$$\prod_k (c_j - \exp(2\pi \mathbf{i} k/n_j) q_{jk}) = 0, \; \prod_{j=1}^{r} c_j = 1,$$

where $q_{jk} = \exp(\tau_{jk})$.

We saw above (Theorem 7.15) that if $\tau_{jk}$'s are formal, the algebra $\mathcal{H}_\tau(\Gamma, H)$ is flat in $\tau$ if $|\Gamma|$ is infinite (i.e., $H$ is Euclidean or hyperbolic). Here is a much stronger non-formal version of this theorem.

**Theorem 7.27.** *The algebra $\mathcal{H}_\tau(\Gamma, H)$ is free as a left module over $R := \mathbb{C}[q_{jk}^{\pm 1}]$ if and only if $\sum_j (1 - 1/m_j) \geq 2$ (i.e., $H$ is Euclidean or hyperbolic).*

*Proof.* Let us consider the Coxeter datum: $m_{ij}$, $i, j = 1, \ldots, r$, such that $m_{i,i+1} := m_i$ ($i \in \mathbb{Z}/r\mathbb{Z}$), and $m_{ij} = \infty$ otherwise. Suppose the corresponding Coxeter group is $W$. Then we can see that $\Gamma = W_+$. Notice that the algebra $\mathcal{H}_\tau(\Gamma, H)$ for genus 0 orbifolds is the algebra $A_+(W)$, i.e., we have $\mathcal{H}_\tau(\Gamma, H) = A_+(W)$.

The condition $\sum_j (1 - 1/m_j) \geq 2$ is equivalent to the condition that $W$ has no finite parabolic subgroups of rank 3. From Theorem 7.26 (ii) and Theorem 7.15, we can see that $A_+(W)$ is free as a left module over $R$. We are done. $\square$

7.15. **Notes.** Section 7.8 follows Section 6 of the paper [EOR]; Cherednik's definition of the double affine Hecke algebra of a root system is from Cherednik's book [Ch]; Sections 7.7 and 7.14 follow the paper [ER]; The other parts of this section follow the paper [E1].

18.735 Double Affine Hecke Algebras in Representation Theory, Combinatorics, Geometry, and Mathematical Physics
Fall 2009