1. **Manifolds**

1.1. **Topological spaces and groups.** Recall that the mathematical notion responsible for describing continuity is that of a **topological space**. Thus, to describe continuous symmetries, we should put this notion together with the notion of a group. This leads to the concept of a **topological group**.

Recall:

• A **topological space** is a set X certain subsets of which (including $\emptyset$ and X) are declared to be **open**, so that an arbitrary union and finite intersection of open sets is open.

• The collection of open sets in X is called **the topology** of X.

• A subset $Z \subset X$ of a topological space X is **closed** if its complement is open.

• If X, Y are topological spaces then the Cartesian product $X \times Y$ has a natural **product topology** in which open sets are (possibly infinite) unions of products $U \times V$, where $U \subset X, V \subset Y$ are open.

• Every subset $Z \subset X$ of a topological space X carries a natural **induced topology**, in which open sets are intersections of open sets in X with Z.

• A map $f : X \to Y$ between topological spaces is **continuous** if for every open set $V \subset Y$, the preimage $f^{-1}(V)$ is open in X.

For example, the open sets of the usual topology of the real line $\mathbb{R}$ are (disjoint) unions of open intervals (a, b), where $-\infty \le a < b \le \infty$.

Definition 1.1. A **topological group** is a group G which is also a topological space, so that the multiplication map $m : G \times G \to G$ and the inversion map $\iota : G \to G$ are continuous.

For example, the group $(\mathbb{R}, +)$ of real numbers with the operation of addition and the usual topology of $\mathbb{R}$ is a topological group, since the functions $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous. Also a subgroup of a topological group is itself a topological group, so another example is rational numbers with addition, $(\mathbb{Q}, +)$. This last example is not a very good model for continuity, however, and shows that general topological groups are not very well behaved. Thus, we will focus on a special class of topological groups called **Lie groups**.

Lie groups are distinguished among topological groups by the property that as topological spaces they belong to a very special class called **topological manifolds.** So we need to start with reviewing this notion.

1.2. **Topological manifolds.** Recall:

• A **neighborhood** of a point $x \in X$ in a topological space X is an open set containing x.

• A **base** for a topological space X is a collection $\mathcal{B}$ of open sets in X such that for every neighborhood U of a point $x \in X$ there exists neighborhood $V \subset U$ of x which belongs to $\mathcal{B}$. Equivalently, every open set in X is a union of members of $\mathcal{B}$.

For example, open intervals form a base of the usual topology of $\mathbb{R}$. Moreover, we may take only intervals whose endpoints have rational coordinates, which gives a *countable* base for $\mathbb{R}$. Also if X, Y are topological spaces with bases $\mathcal{B}_X, \mathcal{B}_Y$ then products $U \times V$, where $U \in \mathcal{B}_X, V \in \mathcal{B}_Y$, form a base of the product topology of $X \times Y$. Thus if X and Y have countable bases, so does $X \times Y$; in particular, $\mathbb{R}^n$ with its usual (product) topology has a countable base (boxes whose vertices have rational coordinates).

• X is **Hausdorff** if any two distinct points have disjoint neighborhoods.

• If X is Hausdorff, we say that a sequence of points $x_n \in X, n \in \mathbb{N}$ **converges** to $x \in X$ as $n \to \infty$ (denoted $x_n \to x$) if every neighborhood of x contains almost all terms of this sequence. Then one also says that the **limit** of x_n is x and writes

$$\lim_{n\to\infty} x_n = x.$$

It is easy to show that the limit is unique when exists. In a Hausdorff space with a countable base, a closed set is one that is closed under taking limits of sequences.

• A Hausdorff space X is **compact** if every open cover $\{U_\alpha, \alpha \in A\}$ of X (i.e., $U_\alpha \subset X$ for all $\alpha \in A$ and $X = \cup_{\alpha\in A} U_\alpha$) has a finite subcover.

• A continuous map $f : X \to Y$ is a **homeomorphism** if it is a bijection and $f^{-1} : Y \to X$ is continuous.

Definition 1.2. A Hausdorff topological space X is said to be an **n-dimensional topological manifold** if it has a countable base and for every $x \in X$ there is a neighborhood $U \subset X$ of x and a continuous map $\phi : U \to \mathbb{R}^n$ such that $\phi : U \to \phi(U)$ is a homeomorphism and $\phi(U) \subset \mathbb{R}^n$ is open.

The second property is often formulated as the condition that X is **locally homeomorphic to** $\mathbb{R}^n$.

It is true (although not immediately obvious) that if a nonempty open set in $\mathbb{R}^n$ is homeomorphic to one in $\mathbb{R}^m$ then $n = m$. Therefore, the number n is uniquely determined by X as long as $X \neq \emptyset$. It is called **the dimension** of X. (By convention, $\emptyset$ is a manifold of any integer dimension).

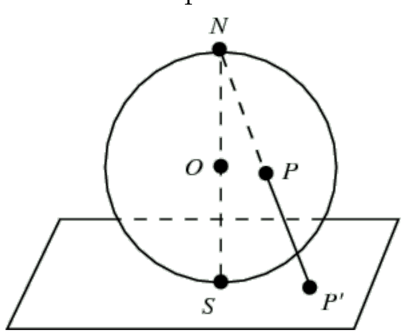
Example 1.3. 1. Obviously $X = \mathbb{R}^n$ is an n-dimensional topological manifold: we can take $U = X$ and $\phi = \mathrm{Id}$.

2. An open subset of a topological manifold is itself a topological manifold of the same dimension.

3. The circle $S^1 \subset \mathbb{R}^2$ defined by the equation $x^2 + y^2 = 1$ is a topological manifold: for example, the point $(1, 0)$ has a neighborhood $U = S^1 \setminus \{(-1, 0)\}$ and a map $\phi : U \to \mathbb{R}$ given by the stereographic projection:

$$\phi(\theta) = \tan(\tfrac{\theta}{2}), \ -\pi < \theta < \pi.$$

and similarly for every other point. More generally, the sphere $S^n \subset \mathbb{R}^{n+1}$ defined by the equation $x_0^2 + ... + x_n^2 = 1$ is a topological manifold, for the same reason. The stereographic projection for the 2-dimensional sphere is shown in the following picture.



4. The curve ∞ is not a manifold, since it is not locally homeomorphic to $\mathbb{R}$ at the self-intersection point (show it!)

A pair (U, ϕ) with the above properties is called a **local chart**. An **atlas** of local charts is a collection of charts $(U_\alpha, \phi_\alpha), \alpha \in A$ such that $\cup_{\alpha \in A} U_\alpha = X$; i.e., $\{U_\alpha, \alpha \in A\}$ is an open cover of X. Thus any topological manifold X admits an atlas labeled by points of X. There are also much smaller atlases. For instance, an open set in $\mathbb{R}^n$ has an atlas with just one chart, while the sphere S^n has an atlas with two charts. Very often X admits an atlas with finitely many charts. For example, if X is compact then there is a finite atlas, since every atlas has a finite subatlas. Moreover, there is always a countable atlas, due to the following lemma:

Lemma 1.4. *If X is a topological space with a countable base then every open cover of X has a countable subcover.*

Proof. Let $\{V_i, i \in \mathbb{N}\}$ be a countable base of X. If $\{U_\alpha\}$ is an open cover of X then for each $x \in X$ pick $\alpha(x)$ such that $x \in V_{i(x)} \subset U_{\alpha(x)}$.

Let $I \subset \mathbb{N}$ be the image of the map i. For each $i \in I$ pick $x \in X$ such that $i(x) = i$ and set $\alpha_i := \alpha(x)$. Then $\{U_{\alpha_i}, i \in I\}$ is a countable subcover of $\{U_\alpha\}$. $\square$

Now let (U, ϕ) and (V, ψ) be two charts such that $V \cap U \neq \emptyset$. Then we have the **transition map**

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$

which is a homeomorphism between open subsets in $\mathbb{R}^n$. For example, consider the atlas of two charts for the circle S^1 (Example 1.3(3)), one missing the point $(-1, 0)$ and the other missing the point $(1, 0)$. Then $\phi(\theta) = \tan(\frac{\theta}{2})$ and $\psi(\theta) = \cot(\frac{\theta}{2})$, $\phi(U \cap V) = \psi(U \cap V) = \mathbb{R} \setminus 0$, and $(\phi \circ \psi^{-1})(x) = \frac{1}{x}$.

1.3. C^k**, real analytic and complex analytic manifolds.** The notion of topological manifold is too general for us, since continuous functions on which it is based in general do not admit a linear approximation. To develop the theory of Lie groups, we need more regularity. So we make the following definition.

Definition 1.5. An atlas on X is said to be **of regularity class** C^k, $1 \le k \le \infty$, if all transition maps between its charts are of class C^k (k times continuously differentiable). An atlas of class C^∞ is called **smooth**. Also an atlas is said to be **real analytic** if all transition maps are real analytic. Finally, if $n = 2m$ is even, so that $\mathbb{R}^n = \mathbb{C}^m$, then an atlas is called **complex analytic** if all its transition maps are complex analytic (i.e., holomorphic).

Example 1.6. The two-chart atlas for the circle S^1 defined by stereographic projections (Example 1.3(3)) is real analytic, since the function $f(x) = \frac{1}{x}$ is analytic. The same applies to the sphere S^n for any n. For example, for S^2 it is easy to see that the transition map $\mathbb{R}^2 \setminus 0 \to \mathbb{R}^2 \setminus 0$ is given by the formula

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Using the complex coordinate $z = x + iy$, we get

$$f(z) = z/|z|^2 = 1/\overline{z}.$$

So this atlas is not complex analytic. But it can be easily made complex analytic by replacing one of the stereographic projections (ϕ or ψ) by its complex conjugate. Then we will have $f(z) = \frac{1}{z}$. On the other hand, it is known (although hard to prove) that S^n does not admit a complex analytic atlas for (even) $n \neq 2, 6$. For $n = 6$ this is a famous conjecture.

Definition 1.7. Two C^k, real analytic, or complex analytic atlases U_α, V_β are said to be **compatible** if the transition maps between U_α and V_β are of the same class (C^k, real analytic, or complex analytic).

It is clear that compatibility is an equivalence relation.

Definition 1.8. A C^k**, real analytic, or complex analytic structure** on a topological manifold X is an equivalence class of C^k, real analytic, or complex analytic atlases. If X is equipped with such a structure, it is said to be a C^k**, real analytic, or complex analytic manifold**. Complex analytic manifolds are also called **complex manifolds**, and a C^∞-manifold is also called **smooth**. A **diffeomorphism** (or **isomorphism**) between such manifolds is a homeomorphism which respects the corresponding classes of atlases.

Remark 1.9. This is really a **structure** and not a **property**. For example, consider $X = \mathbb{C}$ and $Y = D \subset \mathbb{C}$ the open unit disk, with the usual complex coordinate z. It is easy to see that X, Y are isomorphic as real analytic manifolds. But they are not isomorphic as complex analytic manifolds: a complex isomorphism would be a holomorphic function $f : \mathbb{C} \to D$, hence bounded, but by Liouville's theorem any bounded holomorphic function on $\mathbb{C}$ is a constant. Thus we have two different complex structures on $\mathbb{R}^2$ (Riemann showed that there are no others). Also, it is true, but much harder to show, that there are uncountably many different smooth structures on $\mathbb{R}^4$, and there are 28 (oriented) smooth structures on S^7.

Note that the Cartesian product $X \times Y$ of manifolds X, Y is naturally a manifold (of the same regularity type) of dimension $\dim X + \dim Y$.

Exercise 1.10. Let $f_1, ..., f_m$ be functions $\mathbb{R}^n \to \mathbb{R}$ which are C^k or real analytic. Let $X \subset \mathbb{R}^n$ be the set of points P such that $f_i(P) = 0$ for all i and $df_i(P)$ are linearly independent. Use the implicit function theorem to show that X is a topological manifold of dimension $n - m$ and equip it with a natural C^k, respectively real analytic structure. Prove the analogous statement for holomorphic functions $\mathbb{C}^n \to \mathbb{C}$, namely that in this case X is naturally a complex manifold of (complex) dimension $n - m$.

1.4. **Regular functions.** Now let $P \in X$ and (U, ϕ) be a local chart such that $P \in U$ and $\phi(P) = 0$. Such a chart is called a **coordinate chart** around P. In particular, we have **local coordinates** $x_1, ..., x_n : U \to \mathbb{R}$ (or $U \to \mathbb{C}$ for complex manifolds). Note that $x_i(P) = 0$, and $x_i(Q)$ determine Q if $Q \in U$.

Definition 1.11. A **regular function** on an open set $V \subset X$ in a C^k, real analytic, or complex analytic manifold X is a function $f : V \to \mathbb{R}, \mathbb{C}$ such that $f \circ \phi_\alpha^{-1} : \phi_\alpha(V \cap U_\alpha) \to \mathbb{R}, \mathbb{C}$ is of the corresponding regularity class, for some (and then any) atlas (U_α, ϕ_α) defining the corresponding structure on X.[2]

In other words, f is regular if it is expressed as a regular function in local coordinates near every point of V. Clearly, this is independent on the choice of coordinates.

The space (in fact, algebra) of regular functions on V will be denoted by $O(V)$.

Definition 1.12. Let V, U be neighborhoods of $P \in X$. Let us say that $f \in O(V)$, $g \in O(U)$ are **equal near** P if there exists a neighborhood $W \subset U \cap V$ of P such that $f|_W = g|_W$.

It is clear that this is an equivalence relation.

Definition 1.13. A **germ** of a regular function at P is an equivalence class of regular functions defined on neighborhoods of P which are equal near P.

The algebra of germs of regular functions at P is denoted by O_P. Thus we have $O_P = \varinjlim O(U)$, where the direct limit is taken over neighborhoods of P.

1.5. **Tangent spaces.** From now on we will only consider smooth, real analytic and complex analytic manifolds. By a **derivation at** P we will mean a linear map $D : O_P \to \mathbb{R}$ in the smooth and real analytic case and $D : O_P \to \mathbb{C}$ in the complex analytic case, satisfying the Leibniz rule

$$D(fg) = D(f)g(P) + f(P)D(g). \tag{1.1}$$

Note that for any such D we have $D(1) = 0$.

Let $T_P X$ be the space of all such derivations. Thus $T_P X$ is a real vector space for smooth and real analytic manifolds and a complex vector space for complex manifolds.

Lemma 1.14. *Let $x_1, ..., x_n$ be local coordinates at P. Then $T_P X$ has basis $D_1, ..., D_n$, where*

$$D_i(f) := \frac{\partial f}{\partial x_i}(0).$$

[2]More precisely, for C^k and real analytic manifolds regular functions will be assumed real-valued, unless specified otherwise. In the complex analytic case there is, of course, no choice, and regular functions are automatically complex-valued.

Proof. We may assume $X = \mathbb{R}^n$ or $\mathbb{C}^n$, $P = 0$. Clearly, $D_1, ..., D_n$ is a linearly independent set in $T_P X$. Also let $D \in T_P X$, $D(x_i) = a_i$, and consider $D_* := D - \sum_i a_i D_i$. Then $D_*(x_i) = 0$ for all i. Now given a regular function f near 0, for small $x_1, ..., x_n$ by the fundamental theorem of calculus and the chain rule we have:

$$f(x_1, ..., x_n) = f(0) + \int_0^1 \frac{df(tx_1, ..., tx_n)}{dt} dt = f(0) + \sum_{i=1}^n x_i h_i(x_1, ..., x_n),$$

where

$$h_i(x_1, ..., x_n) := \int_0^1 (\partial_i f)(tx_1, ..., tx_n) dt$$

are regular near 0. So by the Leibniz rule $D_*(f) = 0$, hence $D_* = 0$. □

Definition 1.15. The space $T_P X$ is called the **tangent space** to X at P. Elements $v \in T_P X$ are called **tangent vectors** to X at P.

Observe that every tangent vector $v \in T_P X$ defines a derivation $\partial_v : O(U) \to \mathbb{R}, \mathbb{C}$ for every neighborhood U of P, satisfying (1.1). The number $\partial_v f$ is called the **derivative of** f **along** v. For usual curves and surfaces in $\mathbb{R}^3$ these coincide with the familiar notions from calculus.[3]

1.6. Regular maps.

Definition 1.16. A continuous map $F : X \to Y$ between manifolds (of the same regularity class) is **regular** if for any regular function h on an open set $U \subset Y$ the function $h \circ F$ on $F^{-1}(U)$ is regular. In other words, F is regular if it is expressed by regular functions in local coordinates.

It is easy to see that the composition of regular maps is regular, and that a homeomorphism F such that F, F^{-1} are both regular is the same thing as a diffeomorphism (=isomorphism).

Let $F : X \to Y$ be a regular map and $P \in X$. Then we can define the **differential** of F at P, $d_P F$, which is a linear map $T_P X \to T_{F(P)} Y$. Namely, for $f \in O_{F(P)}$ and $v \in T_P X$, the vector $d_P F \cdot v$ is defined by the formula

$$(d_P F \cdot v)(f) := v(f \circ F).$$

The differential of F is also denoted by F_*; namely, for $v \in T_P X$ one writes $dF_P \cdot v = F_* v$.

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[3] Note however that $\partial_v f$ differs from the *directional derivative* $D_v f$ defined in calculus. Namely, $D_v f = \frac{\partial_v f}{|v|}$ (thus defined only for $v \neq 0$) and depends only on the direction of v.

Moreover, if $G : Y \to Z$ is another regular map, then we have the usual **chain rule**,

$$d(G \circ F)_P = dG_{F(P)} \circ dF_P.$$

In particular, if $\gamma : (a, b) \to X$ is a regular **parametrized curve** then for $t \in (a, b)$ we can define the **velocity vector**

$$d_t\gamma \cdot 1 = \gamma'(t) \in T_{\gamma(t)}X$$

(where $1 \in \mathbb{R} = T_t(a, b)$).

1.7. **Submersions and immersions, submanifolds.**

Definition 1.17. A regular map of manifolds $F : X \to Y$ is a **submersion** if $dF_P : T_PX \to T_{F(P)}Y$ is surjective for all $P \in X$.

The following proposition is a version of the implicit function theorem for manifolds.

Proposition 1.18. *If F is a submersion then for any $Q \in Y$, $F^{-1}(Q)$ is a manifold of dimension* $\dim X - \dim Y$.

Proof. This is a local question, so it reduces to the case when X, Y are open subsets in Euclidean spaces. In this case it reduces to Exercise 1.10. □

Definition 1.19. A regular map of manifolds $f : X \to Y$ is an **immersion** if $d_PF : T_PX \to T_{F(P)}Y$ is injective for all $P \in X$.

Example 1.20. The inclusion of the sphere S^n into $\mathbb{R}^{n+1}$ is an immersion. The map $F : S^1 \to \mathbb{R}^2$ given by

$$x(t) = \frac{\cos\theta}{1 + \sin^2\theta},\ y(t) = \frac{\sin\theta\cos\theta}{1 + \sin^2\theta} \tag{1.2}$$

is also an immersion; its image is the lemniscate (shaped as ∞). This shows that an immersion need not be injective. On the other hand, the map $F : \mathbb{R} \to \mathbb{R}^2$ given by $F(t) = (t^2, t^3)$ parametrizing a semicubic parabola $\prec$ is injective, but not an immersion, since $F'(0) = (0, 0)$.

Definition 1.21. An immersion $f : X \to Y$ is an **embedding** if the map $F : X \to F(X)$ is a homeomorphism (where $F(X)$ is equipped with the induced topology from Y). In this case, $F(X) \subset Y$ is said to be an **(embedded) submanifold.**[4]

[4]Recall that a subset Z of a topological space X is called **locally closed** if it is a closed subset in an open subset $U \subset X$. It is clear that embedded submanifolds are locally closed. For this reason they are often called locally closed (embedded) submanifolds.

Example 1.22. The immersion of S^n into $\mathbb{R}^{n+1}$ and of $(0,1)$ into $\mathbb{R}$ are embeddings, but the parametrization of the lemniscate by the circle given by (1.2) is not. The parametrization of the curve ρ by $\mathbb{R}$ is also not an embedding; it is injective but the inverse is not a homeomorphism.

Definition 1.23. An embedding $F : X \to Y$ of manifolds is **closed** if $F(X) \subset Y$ is a closed subset. In this case we say that $F(X)$ is a **closed (embedded) submanifold** of Y.

Example 1.24. The embedding of S^n into $\mathbb{R}^{n+1}$ is closed but of $(0,1)$ into $\mathbb{R}$ is not. Also in Proposition 1.18, $f^{-1}(Q)$ is a closed submanifold of X.