2. Lie groups, I

2.1. The definition of a Lie group.

Definition 2.1. A C^k, real or complex analytic **Lie group** is a manifold G of the same class, with a group structure such that the multiplication map $m : G \times G \to G$ is regular.

Thus, in a Lie group G for any $g \in G$ the left and right translation maps $L_g, R_g : G \to G$, $L_g(x) := gx, R_g(x) := xg$, are diffeomorphisms.

Proposition 2.2. *In a Lie group G, the inversion map $\iota : G \to G$ is a diffeomorphism, and $d\iota_1 = -\mathrm{Id}$.*

Proof. For the first statement it suffices to show that ι is regular near 1, the rest follows by translation. So let us pick a coordinate chart near $1 \in G$ and write the map m in this chart in local coordinates. Note that in these coordinates, $1 \in G$ corresponds to $0 \in \mathbb{R}^n$. Since $m(x, 0) = x$ and $m(0, y) = y$, the linear approximation of $m(x, y)$ at 0 is $x+y$. Thus by the implicit function theorem, the equation $m(x, y) = 0$ is solved near 0 by a regular function $y = \iota(x)$ with $d\iota(0) = -\mathrm{Id}$. This proves the proposition. $\square$

Remark 2.3. A C^0 **Lie group** is a topological group which is a topological manifold. The **Hilbert 5th problem** was to show that any such group is actually a real analytic Lie group (i.e., the regularity class does not matter). This problem is solved by the deep **Gleason-Yamabe theorem**, proved in 1950s. So from now on we will not pay attention to regularity class and consider only real and complex Lie groups.

Note that any complex Lie group of dimension n can be regarded as a real Lie group of dimension $2n$. Also the Cartesian product of real (complex) Lie groups is a real (complex) Lie group.

2.2. Homomorphisms.

Definition 2.4. A **homomorphism of Lie groups** $f : G \to H$ is a group homomorphism which is also a regular map. An **isomorphism of Lie groups** is a homomorphism f which is a group isomorphism, such that $f^{-1} : H \to G$ is regular.

We will see later that the last condition is in fact redundant.

2.3. Examples.

Example 2.5. 1. $(\mathbb{R}^n, +)$ is a real Lie group and $(\mathbb{C}^n, +)$ is a complex Lie group (both n-dimensional).

2. $(\mathbb{R}^\times, \times)$, $(\mathbb{R}_{>0}, \times)$ are real Lie groups, $(\mathbb{C}^\times, \times)$ is a complex Lie group (all 1-dimensional)

3. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a 1-dimensional real Lie group under multiplication of complex numbers.

Note that $\mathbb{R}^\times \cong \mathbb{R}_{>0} \times \mathbb{Z}/2$, $\mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$ as real Lie groups (trigonometric form of a complex number) and $(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \times)$ via $x \mapsto e^x$.

4. The groups of invertible n by n matrices: $GL_n(\mathbb{R})$ is a real Lie group and $GL_n(\mathbb{C})$ is a complex Lie group. These are open sets in the corresponding spaces of all matrices and have dimension n^2.

5. $SU(2)$, the special unitary group of size 2. This is the set of complex 2-by-2 matrices A such that

$$AA^\dagger = \mathbf{1}, \ \det A = 1.$$

So writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix},$$

we get

$$a\bar{a} + b\bar{b} = 1, \ a\bar{c} + b\bar{d} = 0, \ c\bar{c} + d\bar{d} = 1.$$

The second equation implies that $(c, d) = \lambda(-\bar{b}, \bar{a})$. Then we have

$$1 = \det A = ad - bc = \lambda(a\bar{a} + b\bar{b}) = \lambda,$$

so $\lambda = 1$. Thus $SU(2)$ is identified with the set of $(a, b) \in \mathbb{C}^2$ such that $a\bar{a} + b\bar{b} = 1$. Writing $a = x + iy$, $b = z + it$, we have

$$SU(2) = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\}.$$

Thus $SU(2)$ is a 3-dimensional real Lie group which as a manifold is the 3-dimensional sphere $S^3 \subset \mathbb{R}^4$.

6. Any countable group G with **discrete topology** (i.e., such that every set is open) is a (real and complex) Lie group.

2.4. **The connected component of** 1**.** Recall:

• A topological space X is **path-connected** if for any $P, Q \in X$ there is a continuous map $x : [0, 1] \to X$ such that $x(0) = P, x(1) = Q$.

• If X is any topological space, then for $P \in X$ we can define its **path-connected component** to be the set X_P of $Q \in X$ for which there is a continuous map $x : [0, 1] \to X$ such that $x(0) = P, x(1) = Q$ (such x is called **a path connecting** P **and** Q). Then X_P is the largest path-connected subset of X containing P. Clearly, the relation that Q belongs to X_P is an equivalence relation, which splits X into equivalence classes called **path-connected components.** The set of such components is denoted $\pi_0(X)$.

• A topological space X is **connected** if the only subsets of X that are both open and closed are $\emptyset$ and X. For $P \in X$, the **connected component** of X is the union X^P of all connected subsets of X containing P, which is obviously connected itself (so it is the largest connected subset of X containing P). A path-connected space X is always connected but not vice versa (the classic counterexample is the graph of the function $y = \sin(\frac{1}{x})$ together with the interval $[-1, 1]$ of the y-axis); however, a connected *manifold* is path-connected (show it!), so for manifolds the notions of connected component and path-connected component coincide.

• If Y is a topological space, X is a set and $p : Y \to X$ is a surjective map (i.e., $X = Y/\sim$ is the quotient of Y by an equivalence relation) then X acquires a topology called the **quotient topology**, in which open sets are subsets $V \subset X$ such that $p^{-1}(V)$ is open.

Now let G be a real or complex Lie group, and G° the connected component of $1 \in G$. Then the connected component of any $g \in G$ is gG°.

Proposition 2.6. *(i) G° is a normal subgroup of G.*

(ii) $\pi_0(G) = G/G^\circ$ with quotient topology is a discrete and countable group.

Proof. (i) Let $g \in G$, $a \in G^\circ$, and $x : [0, 1] \to G$ be a path connecting 1 to a. Then gxg^{-1} is a path connecting 1 to gag^{-1}, so $gag^{-1} \in G^\circ$, hence G° is normal.

(ii) Since G is a manifold, for any $g \in G$, there is a neighborhood of g contained in $G_g = gG^\circ$. This implies that any coset of G° in G is open, hence G/G° is discrete. Also G/G° is countable since G has a countable base. □

Thus we see that any Lie group is an extension of a discrete countable group by a connected Lie group. This essentially reduces studying Lie groups to studying connected Lie groups. In fact, one can further reduce to simply connected Lie groups, which is done in the next subsections.

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