4. **Homogeneous spaces, Lie group actions**

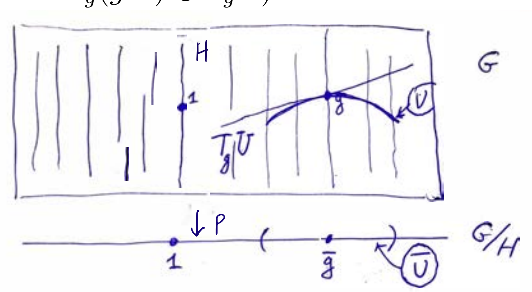
4.1. **Homogeneous spaces.** A regular map of manifolds $p : Y \to X$ is a said to be a **locally trivial fibration** (or **fiber bundle**) with **base** X, **total space** Y and **fiber** being a manifold F if every point $x \in X$ has a neighborhood U such that there is a diffeomorphism $h : U \times F \cong p^{-1}(U)$ with $(p \circ h)(u, f) = u$. In other words, locally p looks like the projection $X \times F \to X$ (the trivial fiber bundle with fiber F over X), but not necessarily globally so. This generalizes the notion of a covering, in which case F is 0-dimensional (discrete).

Theorem 4.1. *(i) Let G be a Lie group of dimension n and $H \subset G$ a closed Lie subgroup of dimension k. Then the* **homogeneous space** *G/H has a natural structure of an $n-k$-dimensional manifold, and the map $p : G \to G/H$ is a locally trivial fibration with fiber H.*

(ii) If moreover H is normal in G then G/H is a Lie group.

(iii) We have a natural isomorphism $T_1(G/H) \cong T_1G/T_1H$.

Proof. Let $\overline{g} \in G/H$ and $g \in p^{-1}(\overline{g})$. Then $gH \subset G$ is an embedded submanifold (image of H under left translation by g). Pick a sufficiently small transversal submanifold U passing through g (i.e., $T_gG = T_g(gH) \oplus T_gU$).



By the inverse function theorem, the set UH is open in G. Let $\overline{U}$ be the image of UH in G/H. Since $p^{-1}(\overline{U}) = UH$ is open, $\overline{U}$ is open in the quotient topology. Also it is clear that $p : U \to \overline{U}$ is a homeomorphism. This defines a local chart near $\overline{g} \in G/H$, and it is easy to check that transition maps between such charts are regular. So G/H acquires the structure of a manifold, which is easily checked to be independent on the choices we made. Also the multiplication map $U \times H \to UH$ is a diffeomorphism, which implies that $p : G \to G/H$ is a locally trivial fibration with fiber H. Finally, we have a surjective linear map $T_gG \to T_{\overline{g}}G/H$ whose kernel is $T_g(gH)$. So in particular for $g = 1$ we get $T_1(G/H) \cong T_1G/T_1H$. This proves all parts of the proposition. □

Recall that a sequence of group homomorphisms $d_i : C^i \to C^{i+1}$ is a **complex** if for all i, $d_i \circ d_{i-1}$ is the trivial homomorphism $C^{i-1} \to C^{i+1}$. (One may consider finite complexes, semi-infinite to the left or to the right, or infinite in both directions). In this case $\mathrm{Im}(d_{i-1}) \subset \mathrm{Ker}(d_i)$ is a subgroup. The i-th **cohomology** $H^i(C^\bullet)$ of the complex $C^\bullet$ is the quotient $\mathrm{Ker}(d_i)/\mathrm{Im}(d_{i-1})$. In general it is just a set but if C^i are abelian groups, it is also an abelian group. Also recall that a complex $C^\bullet$ is called **exact** in the i-th term if $\mathrm{Ker}(d_i) = \mathrm{Im}(d_{i-1})$, i.e., if $H^i(C^\bullet)$ is trivial (consists of one element). A complex exact in all its terms (except possibly first and last, where this condition makes no sense) is called an **exact sequence**.

Corollary 4.2. *Let $H \subset G$ be a closed Lie subgroup.*

(i) If H is connected then the map $p_0 : \pi_0(G) \to \pi_0(G/H)$ is a bijection.

(ii) If also G is connected then there is an exact sequence

$$\pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to 1.$$

Proof. This follows from the theory of covering spaces using that $p : G \to G/H$ is a fibration. □

Exercise 4.3. Fill in the details in the proof of Corollary 4.2.

Remark 4.4. The sequence in Corollary 4.2(ii) is the end portion of the infinite **long exact sequence of homotopy groups of a fibration**,

$$... \to \pi_i(H) \to \pi_i(G) \to \pi_i(G/H) \to \pi_{i-1}(H) \to ...,$$

where $\pi_i(X)$ is the i-th homotopy group of X.

4.2. **Lie subgroups.** We will call the image of an injective immersion of manifolds **an immersed submanifold**; it has a manifold structure coming from the source of the immersion.

Definition 4.5. A **Lie subgroup** of a Lie group G is a subgroup H which is also an *immersed* submanifold (but need not be an *embedded* submanifold, nor a closed subset).

It is clear that in this case H is still a Lie group and the inclusion $H \hookrightarrow G$ is a homomorphism of Lie groups.

Example 4.6. 1. The winding of a torus in Example 3.14(2) realizes $\mathbb{R}$ as a Lie subgroup of $S^1 \times S^1$ which is not closed.

2. Any countable subgroup of G is a 0-dimensional Lie subgroup, but not always a closed one (e.g., $\mathbb{Q} \subset \mathbb{R}$).

Proposition 4.7. *Let $f : G \to K$ be a homomorphism of Lie groups. Then $H := \mathrm{Ker} f$ is a closed normal Lie subgroup in G and $\mathrm{Im} f$ is a Lie subgroup in K, closed if and only if it is an embedded submanifold. In the latter case, we have an isomorphism of Lie groups $G/H \cong \mathrm{Im} f$.*

We will prove Proposition 4.7 in Subsection 9.1.

4.3. **Actions and representations of Lie groups.** Let X be a manifold, G a Lie group, and $a : G \times X \to X$ a set-theoretical left action of G on X.

Definition 4.8. This action is called **regular** if the map a is regular.

From now on, by an action of G on X we will always mean a regular action.

Example 4.9. 1. Any Lie subgroup of $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by linear transformations. Likewise, any Lie subgroup of $GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$.
2. $SO(3)$ acts on S^2 by rotations.

Definition 4.10. A (real analytic) **finite dimensional representation** of a *real* Lie group G is a linear action of G on a finite dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$. Similarly, a (complex analytic) finite dimensional representation of a *complex* Lie group G is a linear action of G on a finite dimensional vector space V over $\mathbb{C}$.

In other words, a representation is a homomorphism of Lie groups $\pi_V : G \to GL(V)$.

Definition 4.11. A **(homo)morphism of representations** (or **intertwining operator**) $A : V \to W$ is a linear map which commutes with the G-action, i.e., $A\pi_V(g) = \pi_W(g)A$, $g \in G$. In particular, if $V = W$, such A is called an **endomorphism** of V.

As usual, an **isomorphism of representations** is an invertible morphism. With these definitions, finite dimensional representations of G form a *category*.

Note also that we have the operations of **dual and tensor product on representations.** Namely, given a representation V of G, we can define its representation on the dual space V^* by

$$\pi_{V^*}(g) = \pi_V(g^{-1})^*,$$

and if W is another representation of G then we can define a representation of G on $V \otimes W$ (the tensor product of vector spaces) by

$$\pi_{V\otimes W}(g) = \pi_V(g) \otimes \pi_W(g).$$

Also if $V \subset W$ is a **subrepresentation** (i.e., a subspace invariant under G) then W/V is also a representation of G, called the **quotient representation**.

4.4. **Orbits and stabilizers.** As in ordinary group theory, if G acts on X and $x \in X$ then we can define the **orbit** $Gx \subset X$ of x as the set of gx, $g \in G$, and the **stabilizer**, or **isotropy group** $G_x \subset G$ to be the group of $g \in G$ such that $gx = x$.

Proposition 4.12. *(The orbit-stabilizer theorem for Lie group actions) The stabilizer $G_x \subset G$ is a closed Lie subgroup, and the natural map $G/G_x \to X$ is an injective immersion whose image is Gx.*

Proposition 4.12 will be proved in Subsection 9.1.

Corollary 4.13. *The orbit $Gx \subset X$ is an immersed submanifold, and we have a natural isomorphism $T_x(Gx) \cong T_1G/T_1G_x$. If Gx is an embedded submanifold then the map $G/G_x \to Gx$ is a diffeomorphism.*

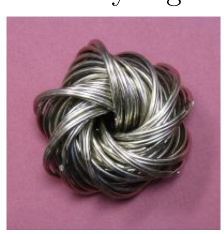
Remark 4.14. Note that Gx need not be closed in X. E.g., let $\mathbb{C}^\times$ act on $\mathbb{C}$ by multiplication. The orbit of 1 is $\mathbb{C}^\times \subset \mathbb{C}$, which is not closed.

Example 4.15. Suppose that G acts on X transitively. Then we get that $X \cong G/G_x$ for any $x \in X$, i.e., X is a **homogeneous space**.

Corollary 4.16. *If G acts transitively on X then the map $p : G \to X$ given by $p(g) = gx$ is a locally trivial fibration with fiber G_x.*

Example 4.17. 1. $SO(3)$ acts transitively on S^2 by rotations, $G_x = S^1 = SO(2)$, so $S^2 = SO(3)/S^1$. Thus $SO(3) = \mathbb{RP}^3$ fibers over S^2 with fiber S^1.

2. $SU(2)$ acts on $S^2 = \mathbb{CP}^1$, and the stabilizer is $S^1 = U(1)$. Thus $SU(2)/S^1 = S^2$, and $SU(2) = S^3$ fibers over S^2 with fiber S^1 (the **Hopf fibration**). Here is D. Richter's keyring model of the Hopf fibration:



3. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{F}_n(\mathbb{K})$ the set of flags $0 \subset V_1 \subset ... \subset V_n = \mathbb{K}^n$ ($\dim V_i = i$). Then $G = GL_n(\mathbb{K})$ acts transitively on $\mathcal{F}_n(\mathbb{K})$ (check it!). Also let $P \in \mathcal{F}_n(\mathbb{K})$ be the flag for which $V_i = \mathbb{K}^i$ is the subspace of vectors whose all coordinates but the first i are zero. Then G_P is the subgroup $B_n(\mathbb{K}) \subset GL_n(\mathbb{K})$ of invertible upper triangular matrices.

Thus $\mathcal{F}_n(\mathbb{K}) = GL_n(\mathbb{K})/B_n(\mathbb{K})$ is a homogeneous space of $GL_n(\mathbb{K})$, in particular, a $\mathbb{K}$-manifold. It is called the **flag manifold**.

4.5. **Left translation, right translation, and adjoint action.** Recall that a Lie group G acts on itself by left translations $L_g(x) = gx$ and right translations $R_{g^{-1}}(x) = xg^{-1}$ (note that both are left actions).

Definition 4.18. The **adjoint action** $\mathrm{Ad}_g : G \to G$ is the action $\mathrm{Ad}_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$; i.e., $\mathrm{Ad}_g(x) = gxg^{-1}$.

Note this is an action by (inner) automorphisms. Also since $\mathrm{Ad}_g(1) = 1$, we have a linear map $d_1\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g} = T_1G$. We will abuse notation and denote this map just by Ad_g. This defines a representation of G on $\mathfrak{g}$ called the **adjoint representation**.