

4. Homogeneous spaces, Lie group actions

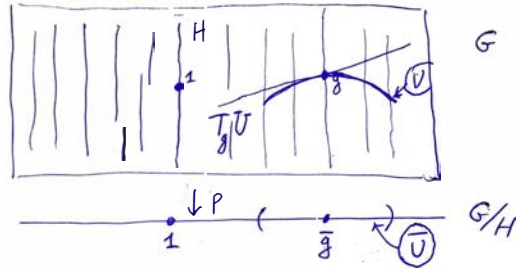
4.1. **Homogeneous spaces.** A regular map of manifolds $p : Y \rightarrow X$ is said to be a **locally trivial fibration** (or **fiber bundle**) with **base** X , **total space** Y and **fiber** being a manifold F if every point $x \in X$ has a neighborhood U such that there is a diffeomorphism $h : U \times F \cong p^{-1}(U)$ with $(p \circ h)(u, f) = u$. In other words, locally p looks like the projection $X \times F \rightarrow X$ (the trivial fiber bundle with fiber F over X), but not necessarily globally so. This generalizes the notion of a covering, in which case F is 0-dimensional (discrete).

Theorem 4.1. (i) Let G be a Lie group of dimension n and $H \subset G$ a closed Lie subgroup of dimension k . Then the **homogeneous space** G/H has a natural structure of an $n - k$ -dimensional manifold, and the map $p : G \rightarrow G/H$ is a locally trivial fibration with fiber H .

(ii) If moreover H is normal in G then G/H is a Lie group.

(iii) We have a natural isomorphism $T_1(G/H) \cong T_1G/T_1H$.

Proof. Let $\bar{g} \in G/H$ and $g \in p^{-1}(\bar{g})$. Then $gH \subset G$ is an embedded submanifold (image of H under left translation by g). Pick a sufficiently small transversal submanifold U passing through g (i.e., $T_gG = T_g(gH) \oplus T_gU$).



By the inverse function theorem, the set UH is open in G . Let \bar{U} be the image of UH in G/H . Since $p^{-1}(\bar{U}) = UH$ is open, \bar{U} is open in the quotient topology. Also it is clear that $p : U \rightarrow \bar{U}$ is a homeomorphism. This defines a local chart near $\bar{g} \in G/H$, and it is easy to check that transition maps between such charts are regular. So G/H acquires the structure of a manifold, which is easily checked to be independent on the choices we made. Also the multiplication map $U \times H \rightarrow UH$ is a diffeomorphism, which implies that $p : G \rightarrow G/H$ is a locally trivial fibration with fiber H . Finally, we have a surjective linear map $T_gG \rightarrow T_{\bar{g}}G/H$ whose kernel is $T_g(gH)$. So in particular for $g = 1$ we get $T_1(G/H) \cong T_1G/T_1H$. This proves all parts of the proposition. \square

Recall that a sequence of group homomorphisms $d_i : C^i \rightarrow C^{i+1}$ is a **complex** if for all i , $d_i \circ d_{i-1}$ is the trivial homomorphism $C_{i-1} \rightarrow C_{i+1}$. (One may consider finite complexes, semi-infinite to the left or to the right, or infinite in both directions). In this case $\text{Im}(d_{i-1}) \subset \text{Ker}(d_i)$ is a subgroup. The i -th **cohomology** $H^i(C^\bullet)$ of the complex C^\bullet is the quotient $\text{Ker}(d_i)/\text{Im}(d_{i-1})$. In general it is just a set but if C^i are abelian groups, it is also an abelian group. Also recall that a complex C^\bullet is called **exact** in the i -th term if $\text{Ker}(d_i) = \text{Im}(d_{i-1})$, i.e., if $H^i(C^\bullet)$ is trivial (consists of one element). A complex exact in all its terms (except possibly first and last, where this condition makes no sense) is called an **exact sequence**.

Corollary 4.2. *Let $H \subset G$ be a closed Lie subgroup.*

(i) *If H is connected then the map $p_0 : \pi_0(G) \rightarrow \pi_0(G/H)$ is a bijection.*

(ii) *If also G is connected then there is an exact sequence*

$$\pi_1(H) \rightarrow \pi_1(G) \rightarrow \pi_1(G/H) \rightarrow 1.$$

Proof. This follows from the theory of covering spaces using that $p : G \rightarrow G/H$ is a fibration. \square

Exercise 4.3. Fill in the details in the proof of Corollary 4.2.

Remark 4.4. The sequence in Corollary 4.2(ii) is the end portion of the infinite **long exact sequence of homotopy groups of a fibration**,

$$\dots \rightarrow \pi_i(H) \rightarrow \pi_i(G) \rightarrow \pi_i(G/H) \rightarrow \pi_{i-1}(H) \rightarrow \dots,$$

where $\pi_i(X)$ is the i -th homotopy group of X .

4.2. Lie subgroups. We will call the image of an injective immersion of manifolds an **immersed submanifold**; it has a manifold structure coming from the source of the immersion.

Definition 4.5. A **Lie subgroup** of a Lie group G is a subgroup H which is also an *immersed* submanifold (but need not be an *embedded* submanifold, nor a closed subset).

It is clear that in this case H is still a Lie group and the inclusion $H \hookrightarrow G$ is a homomorphism of Lie groups.

Example 4.6. 1. The winding of a torus in Example 3.14(2) realizes \mathbb{R} as a Lie subgroup of $S^1 \times S^1$ which is not closed.

2. Any countable subgroup of G is a 0-dimensional Lie subgroup, but not always a closed one (e.g., $\mathbb{Q} \subset \mathbb{R}$).

Proposition 4.7. *Let $f : G \rightarrow K$ be a homomorphism of Lie groups. Then $H := \text{Ker } f$ is a closed normal Lie subgroup in G and $\text{Im } f$ is a Lie subgroup (not necessarily closed) in K if it is an embedded submanifold. In this case we have an isomorphism of Lie groups $G/H \cong \text{Im } f$.*

We will prove Proposition 4.7 in Subsection 9.1.

4.3. Actions and representations of Lie groups. Let X be a manifold, G a Lie group, and $a : G \times X \rightarrow X$ a set-theoretical left action of G on X .

Definition 4.8. This action is called **regular** if the map a is regular.

From now on, by an action of G on X we will always mean a regular action.

Example 4.9. 1. $GL_n(\mathbb{R})$ and any its Lie subgroup acts on \mathbb{R}^n by linear transformations. Likewise, $GL_n(\mathbb{C})$ and any its Lie subgroup acts on \mathbb{C}^n .

2. $SO(3)$ acts on S^2 by rotations.

Definition 4.10. A (real analytic) **finite dimensional representation** of a *real* Lie group G is a linear action of G on a finite dimensional vector space V over \mathbb{R} or \mathbb{C} . Similarly, a (complex analytic) finite dimensional representation of a *complex* Lie group G is a linear action of G on a finite dimensional vector space V over \mathbb{C} .

In other words, a representation is a homomorphism of Lie groups $\pi_V : G \rightarrow GL(V)$.

Definition 4.11. A **(homo)morphism of representations** (or **intertwining operator**) $A : V \rightarrow W$ is a linear map which commutes with the G -action, i.e., $A\pi_V(g) = \pi_W(g)A$, $g \in G$.

As usual, an **isomorphism of representations** is an invertible morphism. With these definitions, finite dimensional representations of G form a *category*.

Note also that we have the operations of **dual and tensor product on representations**. Namely, given a representation V of G , we can define its representation on the dual space V^* by

$$\pi_{V^*}(g) = \pi_V(g^{-1})^*,$$

and if W is another representation of G then we can define a representation of G on $V \otimes W$ (the tensor product of vector spaces) by

$$\pi_{V \otimes W}(g) = \pi_V(g) \otimes \pi_W(g).$$

Also if $V \subset W$ is a **subrepresentation** (i.e., a subspace invariant under G) then W/V is also a representation of G , called the **quotient representation**.

4.4. Orbits and stabilizers. As in ordinary group theory, if G acts on X and $x \in X$ then we can define the **orbit** $Gx \subset X$ of x as the set of gx , $g \in G$, and the **stabilizer**, or **isotropy group** $G_x \subset G$ to be the group of $g \in G$ such that $gx = x$.

Proposition 4.12. *(The orbit-stabilizer theorem for Lie group actions)*
The stabilizer $G_x \subset G$ is a closed Lie subgroup, and the natural map $G/G_x \rightarrow X$ is an injective immersion whose image is Gx .

Proposition 4.12 will be proved in Subsection 9.1.

Corollary 4.13. *The orbit $Gx \subset X$ is an immersed submanifold, and we have a natural isomorphism $T_x(Gx) \cong T_1G/T_1G_x$. If Gx is an embedded submanifold then the map $G/G_x \rightarrow Gx$ is a diffeomorphism.*

Remark 4.14. Note that Gx need not be closed in X . E.g., let \mathbb{C}^\times act on \mathbb{C} by multiplication. The orbit of 1 is $\mathbb{C}^\times \subset \mathbb{C}$, which is not closed.

Example 4.15. Suppose that G acts on X transitively. Then we get that $X \cong G/G_x$ for any $x \in X$, i.e., X is a **homogeneous space**.

Corollary 4.16. *If G acts transitively on X then the map $p: G \rightarrow X$ given by $p(g) = gx$ is a locally trivial fibration with fiber G_x .*

Example 4.17. 1. $SO(3)$ acts transitively on S^2 by rotations, $G_x = S^1 = SO(2)$, so $S^2 = SO(3)/S^1$. Thus $SO(3) = \mathbb{R}P^3$ fibers over S^2 with fiber S^1 .

2. $SU(2)$ acts on $S^2 = \mathbb{C}P^1$, and the stabilizer is $S^1 = U(1)$. Thus $SU(2)/S^1 = S^2$, and $SU(2) = S^3$ fibers over S^2 with fiber S^1 (the **Hopf fibration**). Here is a nice keyring model of the Hopf fibration: <http://homepages.wmich.edu/~drichter/hopffibration.htm>

3. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and $\mathcal{F}_n(\mathbb{K})$ the set of flags $0 \subset V_1 \subset \dots \subset V_n = \mathbb{K}^n$ ($\dim V_i = i$). Then $G = GL_n(\mathbb{K})$ acts transitively on $\mathcal{F}_n(\mathbb{K})$ (check it!). Also let $P \in \mathcal{F}_n(\mathbb{K})$ be the flag for which $V_i = \mathbb{K}^i$ is the subspace of vectors whose all coordinates but the first i are zero. Then G_P is the subgroup $B_n(\mathbb{K}) \subset GL_n(\mathbb{K})$ of invertible upper triangular matrices. Thus $\mathcal{F}_n(\mathbb{K}) = GL_n(\mathbb{K})/B_n(\mathbb{K})$ is a homogeneous space of $GL_n(\mathbb{K})$, in particular, a \mathbb{K} -manifold. It is called the **flag manifold**.

4.5. Left translation, right translation, and adjoint action. Recall that a Lie group G acts on itself by left translations $L_g(x) = gx$ and right translations $R_{g^{-1}}(x) = xg^{-1}$ (note that both are left actions).

Definition 4.18. The **adjoint action** $\text{Ad}_g : G \rightarrow G$ is the action $\text{Ad}_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$; i.e., $\text{Ad}_g(x) = gxg^{-1}$.

Note this is an action by (inner) automorphisms. Also since $\text{Ad}_g(1) = 1$, we have a linear map $d_1\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$, where $\mathfrak{g} = T_1G$. We will abuse notation and denote this map just by Ad_g . This defines a representation of G on \mathfrak{g} called the **adjoint representation**.

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