5. **Tensor fields**

5.1. **A crash course on vector bundles.** Let X be a real manifold. A **vector bundle** on X is, informally speaking, a (locally trivial) fiber bundle on X whose fibers are finite dimensional vector spaces. In other words, it is a family of vector spaces parametrized by $x \in X$ and varying regularly with x. More precisely, we have the following definition.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 5.1. A $\mathbb{K}$-**vector bundle** of rank n on X is a manifold E with a surjective regular map $p : E \to X$ and a $\mathbb{K}$-vector space structure on each fiber $p^{-1}(x)$ such that every $x \in X$ has a neighborhood U admitting a diffeomorphism $h : U \times \mathbb{K}^n \to p^{-1}(U)$ with the following properties:
(i) $(p \circ g_U)(u, v) = u$, and
(ii) the map $g_U : p^{-1}(u) \to u \times \mathbb{K}^n$ is $\mathbb{K}$-linear.

In other words, locally on X, E is isomorphic to $X \times \mathbb{K}^n$, but not necessarily globally so.

As for ordinary fiber bundles, E is called the **total space** and X the **base** of the bundle.

Note that even if X is a complex manifold and $\mathbb{K} = \mathbb{C}$, E need not be a complex manifold.

Definition 5.2. A complex vector bundle $p : E \to X$ on a complex manifold X is said to be **holomorphic** if E is a complex manifold and the diffeomorphisms g_U can be chosen holomorphic.

From now on, unless specified otherwise, all complex vector bundles on complex manifolds we consider will be holomorphic.

It follows from the definition that if $p : E \to X$ is a vector bundle then X has an open cover $\{U_\alpha\}$ such that E trivializes on each U_α, i.e., there is a diffeomorphism $g_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ as above. In this case we have **clutching functions**

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{K})$$

(holomorphic if E is a holomorphic bundle), defined by the formula

$$(g_\alpha \circ g_\beta^{-1})(x, v) = (x, h_{\alpha\beta}(x)v)$$

which satisfy the **consistency conditions**

$$h_{\alpha\beta}(x) = h_{\beta\alpha}(x)^{-1}$$

and

$$h_{\alpha\beta}(x) \circ h_{\beta\gamma}(x) = h_{\alpha\gamma}(x)$$

for $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Moreover, the bundle can be reconstructed from this data, starting from the disjoint union $\sqcup_\alpha U_\alpha \times \mathbb{K}^n$ and identifying (gluing) points according to

$$h_{\alpha\beta} : (x, v) \in U_\beta \times \mathbb{K}^n \sim (x, h_{\alpha\beta}(x)v) \in U_\alpha \times \mathbb{K}^n.$$

The consistency conditions ensure that the relation $\sim$ is symmetric and transitive, so it is an equivalence relation, and we define E to be the space of equivalence classes with the quotient topology. Then E has a natural structure of a vector bundle on X.

This can also be used for constructing vector bundles. Namely, the above construction defines a $\mathbb{K}$-vector bundle on X once we are given a cover $\{U_\alpha\}$ on X and a collection of clutching functions

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{K})$$

satisfying the consistency conditions.

Remark 5.3. All this works more generally for non-linear fiber bundles if we drop the linearity conditions along fibers.

Example 5.4. 1. The **trivial bundle** $p : E = X \times \mathbb{K}^n \to X$, $p(x, v) = x$.

2. The **tangent bundle** is the vector bundle $p : TX \to X$ constructed as follows. For the open cover we take an atlas of charts (U_α, ϕ_α) with transition maps

$$\theta_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta),$$

and we set

$$h_{\alpha\beta}(x) := d_{\phi_\beta(x)}\theta_{\alpha\beta}.$$

(Check that these maps satisfy consistency conditions!)

Thus the tangent bundle TX is a vector bundle of rank $\dim X$ whose fiber $p^{-1}(x)$ is naturally the tangent space $T_x X$ (indeed, the tangent vectors transform under coordinate changes exactly by multiplication by $h_{\alpha\beta}(x)$). In other words, it formalizes the idea of "the tangent space $T_x X$ varying smoothly with $x \in X$".

Definition 5.5. A **section** of a map $p : E \to X$ is a map $s : X \to E$ such that $p \circ s = \mathrm{Id}_x$.

Example 5.6. If $p : X \times Y = E \to X$, $p(x, y) = x$ is the trivial bundle then a section $s : X \to E$ is given by $s(x) = (x, f(x))$ where $y = f(x)$ is a function $X \to Y$, and the image of s is the graph of f. So the notion of a section is a generalization of the notion of a function.

In particular, we may consider sections of a vector bundle $p : E \to X$ over an open set $U \subset X$. These sections form a vector space denoted $\Gamma(U, E)$.

Exercise 5.7. Show that a vector bundle $p : E \to X$ is trivial (i.e., globally isomorphic to $X \times \mathbb{K}^n$) if and only if it admits sections $s_1, ..., s_n$ which form a basis in every fiber $p^{-1}(x)$.

5.2. **Vector fields.**

Definition 5.8. A **vector field** on X is a section of the tangent bundle TX.

Thus in local coordinates a vector field looks like

$$\mathbf{v} = \sum_i v_i \frac{\partial}{\partial x_i},$$

$v_i = v_i(\mathbf{x})$, and if $x_i \mapsto x_i'$ is a change of local coordinates then the expression for $\mathbf{v}$ in the new coordinates is

$$\mathbf{v} = \sum_i v_i' \frac{\partial}{\partial x_i'}$$

where

$$v_i' = \sum_j \frac{\partial x_i'}{\partial x_j} v_j,$$

i.e., the clutching function is the **Jacobi matrix** of the change of variable. Thus, every vector field $\mathbf{v}$ on X defines a derivation of the algebra $O(U)$ for every open set $U \subset X$ compatible with restriction maps $O(U) \to O(V)$ for $V \subset U$; in particular, a derivation $O_x \to O_x$ for all $x \in X$. Conversely, it is easy to see that such a collection of derivations gives rise to a vector field, so this is really the same thing.

A manifold X is called **parallelizable** if its tangent bundle is trivial. By Exercise 5.7, this is equivalent to having a collection of vector fields $\mathbf{v}_1, ..., \mathbf{v}_n$ which form a basis in every tangent space (such a collection is called a **frame**). For example, the circle S^1 and hence the torus $S^1 \times S^1$ are parallelizable. On the other hand, the sphere S^2 is not parallelizable, since it does not even have a single nowhere vanishing vector field (the **Hairy Ball theorem**, or **Hedgehog theorem**). The same is true for any even-dimensional sphere S^{2m}, $m \geq 1$.

5.3. **Tensor fields, differential forms.** Since vector bundles are basically just smooth families of vector spaces varying over some base manifold X, we can do with them the same things we can do with vector spaces - duals, tensor products, symmetric and exterior powers, etc. E.g., the **cotangent bundle** T^*X is dual to the tangent bundle TX.

More generally, we make the following definition.

Definition 5.9. A **tensor field** of rank (k, m) on a manifold X is a section of the tensor product $(TX)^{\otimes k} \otimes (T^*X)^{\otimes m}$.

For example, a tensor field of rank $(1, 0)$ is a vector field. Also, a skew-symmetric tensor field of rank $(0, m)$ is called a **differential m-form** on X. In other words, a differential m-form is a section of the vector bundle $\Lambda^m T^*X$.

For instance, if $f \in O(X)$ then we have a differential 1-form df on X, called **the differential of** f (indeed, recall that $d_x f : T_x X \to \mathbb{K}$). A general 1-form can therefore be written in local coordinates as

$$\omega = \sum_i a_i dx_i.$$

where $a_i = a_i(\mathbf{x})$. If coordinates are changed as $x_i \mapsto x_i'$, then in new coordinates

$$\omega = \sum_i a_i' dx_i'$$

where

$$a_i' = \sum_j \frac{\partial x_j}{\partial x_i'} a_j.$$

Thus the clutching function is the **inverse of the Jacobi matrix** of the change of variable. For instance,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

More generally, a differential m-form in local coordinates looks like

$$\omega = \sum_{1 \le i_1 < \ldots < i_m \le n} a_{i_1 \ldots i_m}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_m}.$$

5.4. **Left and right invariant tensor fields on Lie groups.** Note that if a Lie group G acts on a manifold X, then it automatically acts on the tangent bundle TX and thus on vector and, more generally, tensor fields on X. In particular, G acts on tensor fields on itself by left and right translations; we will denote this action by L_g and R_g, respectively. We say that a tensor field T on G is **left invariant** if $L_g T = T$ for all $g \in G$, and **right invariant** if $R_g T = T$ for all $g \in G$.

Proposition 5.10. *(i) For any $\tau \in \mathfrak{g}^{\otimes k} \otimes \mathfrak{g}^{*\otimes m}$ there exists a unique left invariant tensor field $\mathbf{L}_\tau$ and a unique right invariant tensor field*

$\mathbf{R}_\tau$ *whose value at* 1 *is* τ. *Thus, the spaces of such tensor fields are naturally isomorphic to* $\mathfrak{g}^{\otimes k} \otimes \mathfrak{g}^{*\otimes m}$.

(ii) $\mathbf{L}_\tau$ *is also right invariant iff* $\mathbf{R}_\tau$ *is also left invariant iff* τ *is invariant under the adjoint representation* Ad_g.

Proof. We only prove (i). Consider the tensor fields $\mathbf{L}_\tau(g) := L_g\tau, \mathbf{R}_\tau(g) := R_{g^{-1}}\tau$ (i.e., we "spread" τ from $1 \in G$ to other points $g \in G$ by left/right translations). By construction, $R_{g^{-1}}\tau$ is right invariant, while $L_g\tau$ is left invariant, both with value τ at 1, and it is clear that these are unique. □

Exercise 5.11. Prove Proposition 5.10(ii).

Corollary 5.12. *A Lie group is parallelizable.*

Proof. Given a basis $e_1, ..., e_n$ of $\mathfrak{g} = T_1G$, the vector fields $L_ge_1, ..., L_ge_n$ form a frame. □

Remark 5.13. In particular, S^1 and $SU(2) = S^3$ are parallelizable. It turns out that S^n for $n \geq 1$ is parallelizable if and only if $n = 1, 3, 7$ (a deep theorem in differential topology). So spheres of other dimensions don't admit a Lie group structure. The sphere S^7 does not admit one either, although it admits a weaker structure of a "homotopy Lie group", or H-space (arising from octonions) which suffices for parallelizability. Thus the only spheres admitting a Lie group structure are $S^0 = \{1, -1\}$, S^1 and S^3. This result is fairly elementary and will be proved in Section 46.