6. **Classical Lie groups**

6.1. **First examples of classical groups.** Roughly speaking, **classical groups** are groups of matrices arising from linear algebra. More precisely, classical groups are the following subgroups of the **general linear group** $GL_n(\mathbb{K})$: $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$ (the **special linear group**), $O_n(\mathbb{K})$, $SO_n(\mathbb{K})$, $Sp_{2n}(\mathbb{K})$, $O(p,q)$, $SO(p,q)$, $U(p,q)$, $SU(p,q)$, $Sp(2p,2q) := Sp_{2n}(\mathbb{C}) \cap U(2p,2q)$ for $p+q=n$ (and also some others we'll consider later).

Namely,

- The **orthogonal group** $O_n(\mathbb{K})$ is the group of matrices preserving the nondegenerate quadratic form in n variables, $Q = x_1^2 + ... + x_n^2$ (or, equivalently, the corresponding bilinear form $x_1y_1 + ... + x_ny_n$);
- The **symplectic group** $Sp_{2n}(\mathbb{K})$ is the group of matrices preserving a nondegenerate skew-symmetric form in $2n$ variables;
- The **pseudo-orthogonal group** $O(p,q)$, $p+q=n$ is the group of real matrices preserving a nondegenerate quadratic form of signature (p,q), $Q = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2$ (or, equivalently, the corresponding bilinear form);
- The **pseudo-unitary group** $U(p,q)$, $p+q=n$ is the group of complex matrices preserving a nondegenerate Hermitian quadratic form of signature (p,q), $Q = |x_1|^2 + ... + |x_p|^2 - |x_{p+1}|^2 - ... - |x_n|^2$ (or, equivalently, the corresponding sesquilinear form);
- The **special pseudo-orthogonal, pseudo-unitary, and orthogonal groups** $SO(p,q) \subset O(p,q)$, $SU(p,q) \subset U(p,q)$, $SO_n \subset O_n$ are the subgroups of matrices of determinant 1.

Note that the groups don't change under switching p,q and that $(S)O_n(\mathbb{R}) = (S)O(n,0)$; it is also denoted $(S)O(n)$. Also $(S)U(n,0)$ is denoted by $(S)U(n)$.

Exercise 6.1. Show that the special (pseudo)orthogonal groups are index 2 subgroups of the (pseudo)orthogonal groups.

Let us show that they are all Lie groups. For this purpose we'll use the **exponential map** for matrices. Namely, recall from linear algebra that we have an analytic function $\exp : \mathfrak{gl}_n(\mathbb{K}) \to GL_n(\mathbb{K})$ given by the formula

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

and the matrix-valued analytic function log near $1 \in GL_n(\mathbb{K})$,

$$\log(A) = -\sum_{n=1}^{\infty} \frac{(1-A)^n}{n}.$$

Namely, this is well defined if the spectral radius of $1-A$ is < 1 (i.e., all eigenvalues are in the open unit disk). These maps have the following properties:

1. They are mutually inverse.
2. They are conjugation-invariant.
3. $d\exp_0 = d\log_1 = \mathrm{Id}$.
4. If $xy = yx$ then $\exp(x+y) = \exp(x)\exp(y)$. If $XY = YX$ then $\log(XY) = \log(X) + \log(Y)$ (for X, Y sufficiently close to 1).
5. For $x \in \mathfrak{gl}_n(\mathbb{K})$ the map $t \mapsto \exp(tx)$ is a homomorphism of Lie groups $\mathbb{K} \to GL_n(\mathbb{K})$.
6. $\det\exp(a) = \exp(\mathrm{Tr}\ a)$, $\log(\det A) = \mathrm{Tr}(\log A)$.

Now we can look at classical groups and see what happens to the equations defining them when we apply log.

1. $G = SL_n(\mathbb{K})$. We already showed that it is a Lie group in Example 3.14 but let us re-do it by a different method. The group G is defined by the equation $\det A = 1$. So for A close to 1 we have $\log(\det A) = 0$, i.e., $\mathrm{Tr}\log(A) = 0$. So $\log(A) \in \mathfrak{sl}_n(\mathbb{K}) = \mathfrak{g}$, the space of matrices with trace 0. This defines a local chart near $1 \in G$, showing that G is a manifold, hence a Lie group (namely, local charts near other points are obtained by translation).

2. $G = O_n(\mathbb{K})$. The equation is $A^T = A^{-1}$, thus $\log(A)^T = -\log(A)$, so $\log(A) \in \mathfrak{so}_n(\mathbb{K}) = \mathfrak{g}$, the space of skew-symmetric matrices.

3. $G = U(n)$. The equation is $\overline{A}^T = A^{-1}$, thus $\overline{\log(A)}^T = -\log(A)$, so $\log(A) \in \mathfrak{u}_n = \mathfrak{g}$, the space of skew-Hermitian matrices.

Exercise 6.2. Do the same for all classical groups listed above.

We obtain

Proposition 6.3. *Every classical group G from the above list is a Lie group, with $\mathfrak{g} = T_1 G \subset \mathfrak{gl}_n(\mathbb{K})$. Moreover, if $\mathfrak{u} \subset \mathfrak{gl}_n(\mathbb{K})$ is a small neighborhood of 0 and $U = \exp(\mathfrak{u})$ then* exp *and* log *define mutually inverse diffeomorphisms between $\mathfrak{u} \cap \mathfrak{g}$ and $U \cap G$.*

Exercise 6.4. Which of these groups are complex Lie groups?

Exercise 6.5. Use this proposition to compute the dimensions of classical groups: $\dim SL_n = n^2 - 1$, $\dim O_n = n(n-1)/2$, $\dim Sp_{2n} = n(2n+1)$, $\dim SU_n = n^2 - 1$, etc. (Note that for complex groups we give the dimension over $\mathbb{C}$).

6.2. **Quaternions.** An important role in the theory of Lie groups is played by the **algebra of quaternions**, which is the only noncommutative finite dimensional division algebra over $\mathbb{R}$, discovered in the 19th century by W. R. Hamilton.

Definition 6.6. The **algebra of quaternions** is the $\mathbb{R}$-algebra with basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

This algebra is associative but not commutative.

Given a quaternion

$$\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},\ a, b, c, d \in \mathbb{R},$$

we define the **conjugate quaternion** by the formula

$$\overline{\mathbf{q}} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

Thus

$$\mathbf{q}\overline{\mathbf{q}} = |\mathbf{q}|^2 = a^2 + b^2 + c^2 + d^2 \in \mathbb{R},$$

where $|\mathbf{q}|$ is the length of $\mathbf{q}$ as a vector in $\mathbb{R}^4$. So if $\mathbf{q} \neq 0$ then it is invertible and

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}.$$

Thus $\mathbb{H}$ is a **division algebra** (i.e., a skew-field). One can show that the only finite dimensional associative division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. (See Exercise 6.9).

In particular, we can do linear algebra over $\mathbb{H}$ in almost the same way as we do over ordinary fields. Namely, every (left or right) module over $\mathbb{H}$ is free and has a basis; such a module is called a (left or right) **quaternionic vector space**. In particular, any (say, right) quaternionic vector space of dimension n (i.e., with basis of n elements) is isomorphic to $\mathbb{H}^n$. Moreover, $\mathbb{H}$-linear maps between such spaces are given by left multiplication by quaternionic matrices. Finally, it is easy to see that Gaussian elimination works the same way as over ordinary fields; in particular, every invertible square matrix over $\mathbb{H}$ is a product of elementary matrices of the form $1 + (\mathbf{q} - 1)E_{ii}$ and $1 + \mathbf{q}E_{ij}$, $i \neq j$, where $\mathbf{q} \in \mathbb{H}$ is nonzero.

Also it is easy to show that

$$\overline{\mathbf{q}_1\mathbf{q}_2} = \overline{\mathbf{q}_2\mathbf{q}_1},\ |\mathbf{q}_1\mathbf{q}_2| = |\mathbf{q}_1| \cdot |\mathbf{q}_2|$$

(check this!). So quaternions are similar to complex numbers, except they are non-commutative. Finally, note that $\mathbb{H}$ contains a copy of $\mathbb{C}$ spanned by $1, \mathbf{i}$; however, this does not make $\mathbb{H}$ as $\mathbb{C}$-algebra since $\mathbf{i}$ is not a central element.

Proposition 6.7. *The group of unit quaternions* $\{\mathbf{q} \in \mathbb{H} : |\mathbf{q}| = 1\}$ *under multiplication is isomorphic to* $SU(2)$ *as a Lie group.*

Proof. We can realize $\mathbb{H}$ as $\mathbb{C}^2$, where $\mathbb{C} \subset \mathbb{H}$ is spanned by $1, \mathbf{i}$; namely, $(z_1, z_2) \mapsto z_1 + \mathbf{j}z_2$. Then left multiplication by quaternions on $\mathbb{H} = \mathbb{C}^2$ commutes with right multiplication by $\mathbb{C}$, i.e., is $\mathbb{C}$-linear. So it is given by complex 2-by-2 matrices. It is easy to compute that the corresponding matrix is

$$z_1 + z_2\mathbf{j} \mapsto \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix},$$

and we showed in Example 2.3(5) that such matrices (with $|z_1|^2+|z_2|^2 = 1$) are exactly the matrices from $SU(2)$. □

This is another way to see that $SU(2) \cong S^3$ as a manifold (since the set of unit quaternions is manifestly S^3).

Corollary 6.8. *The map* $\mathbf{q} \mapsto (\frac{\mathbf{q}}{|\mathbf{q}|}, |\mathbf{q}|)$ *is an isomorphism of Lie groups* $\mathbb{H}^\times \cong SU(2) \times \mathbb{R}_{>0}$.

This is the quaternionic analog of the trigonometric form of complex numbers, except the "phase" factor $\frac{\mathbf{q}}{|\mathbf{q}|}$ is now not in S^1 but in $S^3 = SU(2)$.

Exercise 6.9. Let D be a finite dimensional division algebra over $\mathbb{R}$.

(i) Show that if D is commutative then $D = \mathbb{R}$ or $D = \mathbb{C}$.

(ii) Assume that D is not commutative. Take $\mathbf{q} \in D$, $\mathbf{q} \notin \mathbb{R}$. Show that there exist $a, b \in \mathbb{R}$ such that $\mathbf{i} := a + b\mathbf{q}$ satisfies $\mathbf{i}^2 = -1$.

(iii) Decompose D into the eigenspaces $D_\pm$ of the operator of conjugation by $\mathbf{i}$ with eigenvalues ± 1 and show that $1, \mathbf{i}$ is a basis of D_+, i.e., $D_+ \cong \mathbb{C}$.

(iv) Pick $\mathbf{q} \in D_-$, $\mathbf{q} \neq 0$, and show that $D_- = D_+\mathbf{q}$, so $\{1, \mathbf{i}, \mathbf{q}, \mathbf{iq}\}$ is a basis of D over $\mathbb{R}$. Deduce that $\mathbf{q}^2$ is a central element of D.

(v) Conclude that $\mathbf{q}^2 = -\lambda$ where $\lambda \in \mathbb{R}_{>0}$ and deduce that $D \cong \mathbb{H}$.

6.3. **More classical groups.** Now we can define a new classical group $GL_n(\mathbb{H})$, a real Lie group of dimension $4n^2$, called the **quaternionic general linear group**. For example, as we just showed, $GL_1(\mathbb{H}) = \mathbb{H}^\times \cong SU(2) \times \mathbb{R}_{>0}$.

For $A \in GL_n(\mathbb{H})$, let $\det A$ be the determinant of A as a linear operator on $\mathbb{C}^{2n} = \mathbb{H}^n$.

Lemma 6.10. *We have* $\det A > 0$.

Proof. For $n = 1$, $A = \mathbf{q} \in \mathbb{H}^\times$ and $\det \mathbf{q} = |\mathbf{q}|^2 > 0$. It follows that $\det(1+(\mathbf{q}-1)E_{ii}) = |\mathbf{q}|^2 > 0$. Also it is easy to see that $\det(1+\mathbf{q}E_{ij}) = 1$ for $i \neq j$. It then follows by Gaussian elimination that for any A we have $\det(A) > 0$. □

Let $SL_n(\mathbb{H}) \subset GL_n(\mathbb{H})$ be the subgroup of matrices A with $\det A = 1$, called the **quaternionic special linear group**.

Exercise 6.11. Show that $SL_n(\mathbb{H}) \subset GL_n(\mathbb{H})$ is a normal subgroup, and $GL_n(\mathbb{H}) \cong SL_n(\mathbb{H}) \times \mathbb{R}_{>0}$.

Thus $SL_n(\mathbb{H})$ is a real Lie group of dimension $4n^2 - 1$.

We can also define groups of quaternionic matrices preserving various sesquilinear forms. Namely, let $V \cong \mathbb{H}^n$ be a right quaternionic vector space.

Definition 6.12. A **sesquilinear form** on V is a biadditive function $(,) : V \times V \to \mathbb{H}$ such that

$$(\mathbf{x}\alpha, \mathbf{y}\beta) = \overline{\alpha}(\mathbf{x}, \mathbf{y})\beta,\ \mathbf{x}, \mathbf{y} \in V,\ \alpha, \beta \in \mathbb{H}.$$

Such a form is called **Hermitian** if $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ and **skew-Hermitian** if $(\mathbf{x}, \mathbf{y}) = -\overline{(\mathbf{y}, \mathbf{x})}$.

Note that the order of factors is important here!

Proposition 6.13. *(i) Every nondegenerate Hermitian form on V in some basis takes the form*

$$(\mathbf{x}, \mathbf{y}) = \overline{x_1}y_1 + ... + \overline{x_p}y_p - \overline{x_{p+1}}y_{p+1} - ... - \overline{x_n}y_n$$

for a unique pair (p, q) with $p + q = n$.

(ii) Every nondegenerate skew-Hermitian form on V in some basis takes the form

$$(\mathbf{x}, \mathbf{y}) = \overline{x_1}\mathbf{j}y_1 + ... + \overline{x_n}\mathbf{j}y_n.$$

Exercise 6.14. Prove Proposition 6.13.

In (i), the pair (p, q) is called the **signature** of the quaternionic Hermitian form.

Exercise 6.15. Show that a nondegenerate quaternionic Hermitian form of signature (p, q) can be written as

$$(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}, \mathbf{y}) + \mathbf{j}B_2(\mathbf{x}, \mathbf{y}),$$

with B_1, B_2 taking values in $\mathbb{C} = \mathbb{R} + \mathbb{R}\mathbf{i} \subset \mathbb{H}$, where B_1 is a usual nondegenerate Hermitian form of signature $(2p, 2q)$ and B_2 is a nondegenerate skew-symmetric bilinear form on V as a ($2n$-dimensional) $\mathbb{C}$-vector space. Show that $B_2(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}\mathbf{j}, \mathbf{y})$. Deduce that any complex linear transformation preserving B_1 and B_2 is $\mathbb{H}$-linear.

Thus the group of symmetries of a nondegenerate quaternionic Hermitian form of signature (p, q) is $Sp(2p, 2q) = Sp_{2n}(\mathbb{C}) \cap U(2p, 2q)$. It is called the **quaternionic pseudo-unitary group**.

One also sometimes uses the notation $U(p,q,\mathbb{R}) = O(p,q)$, $U(p,q,\mathbb{C}) = U(p,q)$, $U(p,q,\mathbb{H}) = Sp(2p,2q)$, and $U(n,0,\mathbb{K}) = U(n,\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Exercise 6.16. Show that a nondegenerate quaternionic skew-Hermitian form can be written as

$$(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}, \mathbf{y}) + \mathbf{j}B_2(\mathbf{x}, \mathbf{y}),$$

with B_1, B_2 taking values in $\mathbb{C} = \mathbb{R} + \mathbb{R}\mathbf{i} \subset \mathbb{H}$, where B_1 is an ordinary skew-Hermitian form, while B_2 is a symmetric bilinear form (both nondegenerate). Show that $B_2(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}\mathbf{j}, \mathbf{y})$. Deduce that any complex linear transformation preserving B_1 and B_2 is $\mathbb{H}$-linear. Also show that the signature of the Hermitian form iB_1 is necessarily (n,n).

Thus the group of symmetries of a nondegenerate quaternionic skew-Hermitian form is $O_{2n}(\mathbb{C}) \cap U(n,n)$. This group is denoted by $O^*(2n)$ and called the **quaternionic orthogonal group**. There is also the subgroup $SO^*(2n) \subset O^*(2n)$ of matrices of determinant 1 (having index 2).

All of these groups are Lie groups, which is shown similarly to Subsection 6.1, using the exponential map.

Exercise 6.17. Compute the dimensions of all classical groups introduced above.