9. **Fundamental theorems of Lie theory**

9.1. **Proofs of Theorem 3.13, Proposition 4.12, Proposition 4.7.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and X be a manifold with an action $a : G \times X \to X$. Then for any $z \in \mathfrak{g}$ we have a vector field $a_*(z)$ on X given by

$$(a_*(z)f)(x) = \frac{d}{dt}|_{t=0} f(\exp(-tz)x),$$

where $t \in \mathbb{R}$, $f \in O(U)$ for some open set $U \subset X$ and $x \in U$.

Proposition 9.1. *The map a_* is linear and we have*

$$a_*([z, w]) = [a_*(z), a_*(w)].$$

In other words, the map $a_* : \mathfrak{g} \to \mathrm{Vect}(X)$ is a homomorphism of Lie algebras.

Exercise 9.2. Prove Proposition 9.1.

This motivates the following definition.

Definition 9.3. An action of a Lie algebra $\mathfrak{g}$ on a manifold X is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{Vect}(X)$.

Thus an action of a Lie group G on X induces an action of the Lie algebra $\mathfrak{g} = \mathrm{Lie}G$ on X.

Now let $x \in X$. Then we have a linear map $a_{*x} : \mathfrak{g} \to T_xX$ given by $a_{*x}(z) := a_*(z)(x)$.

Theorem 9.4. *(i) The stabilizer G_x is a closed subgroup of G with Lie algebra*

$$\mathfrak{g}_x := \mathrm{Ker}(a_{*x}).$$

(ii) The map $G/G_x \to X$ given by $g \mapsto gx$ is an immersion. So the orbit Gx is an immersed submanifold of X, and

$$T_x(Gx) \cong \mathrm{Im}(a_{*x}) \cong \mathfrak{g}/\mathfrak{g}_x.$$

Part (i) of Theorem 9.4 is the promised weaker version of Theorem 3.13 sufficient for our purposes. Also, part (ii) implies Proposition 4.12.

Proof. (i) It is clear that G_x is closed in G, but we need to show it is a Lie subgroup and compute its Lie algebra.[7] It suffices to show that for some neighborhood U of 1 in G, $U \cap G_x$ is a (closed) submanifold of U such that $T_1(U \cap G_x) = \mathfrak{g}_x$.

Note that $\mathfrak{g}_x \subset \mathfrak{g}$ is a Lie subalgebra, since the commutator of vector fields vanishing at x also vanishes at x (by the formula for commutator

[7]Although we claimed in Theorem 3.13 that a closed subgroup of a Lie group is always a Lie subgroup, we did not prove it, so we need to prove it in this case.

in local coordinates). Also, for any $z \in \mathfrak{g}_x$, $\exp(tz)x$ is a solution of the ODE $\gamma'(t) = a_{*\gamma(t)}(z)$ with initial condition $\gamma(0) = x$, and $\gamma(t) = x$ is such a solution, so by uniqueness of ODE solutions $\exp(tz)x = x$, thus $\exp(tz) \in G_x$.

Now choose a complement $\mathfrak{u}$ of $\mathfrak{g}_x$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{u}$. Then $a_{*x} : \mathfrak{u} \to T_xX$ is injective. By the implicit function theorem, the map $\mathfrak{u} \to X$ given by $u \mapsto \exp(u)x$ is injective for small u, so $\exp(u) \in G_x$ for small $u \in \mathfrak{u}$ if and only if $u = 0$.

But in a small neighborhood U of 1 in G, any element g can be uniquely written as $g = \exp(u)\exp(z)$, where $u \in \mathfrak{u}$ and $z \in \mathfrak{g}_x$. So we see that $g \in G_x$ iff $u = 0$, i.e., $\log(g) \in \mathfrak{g}_x$. This shows that $U \cap G_x$ coincides with $U \cap \exp(\mathfrak{g}_x)$, as desired.

(ii) The same proof shows that we have an isomorphism $T_1(G/G_x) \cong \mathfrak{g}/\mathfrak{g}_x = \mathfrak{u}$, so the injectivity of $a_{*x} : \mathfrak{u} \to T_xX$ implies that the map $G/G_x \to X$ given by $g \mapsto gx$ is an immersion, as claimed. □

Corollary 9.5. *(Proposition 4.7) Let $\phi : G \to K$ be a morphism of Lie groups and $\phi_* : \mathrm{Lie}G \to \mathrm{Lie}K$ be the corresponding morphism of Lie algebras. Then $H := \mathrm{Ker}(\phi)$ is a closed normal Lie subgroup with Lie algebra $\mathfrak{h} := \mathrm{Ker}(\phi_*)$, and the map $\overline{\phi} : G/H \to K$ is an immersion. Moreover, if $\mathrm{Im}\overline{\phi}$ is a submanifold of K then it is a closed Lie subgroup, and we have an isomorphism of Lie groups $\overline{\phi} : G/H \cong \mathrm{Im}\overline{\phi}$.*

Proof. Apply Theorem 9.4 to the action of G on $X = K$ via $g \circ k = \phi(g)k$, and take $x = 1$. □

Corollary 9.6. *Let V be a finite dimensional representation of a Lie group G, and $v \in V$. Then the stabilizer G_v is a closed Lie subgroup of G with Lie algebra $\mathfrak{g}_v := \{z \in \mathfrak{g} : zv = 0\}$.*

Example 9.7. Let A be a finite dimensional algebra (not necessarily associative, e.g. a Lie algebra). Then the group $G = \mathrm{Aut}(A) \subset GL(A)$ is a closed Lie subgroup with Lie algebra $\mathrm{Der}(A) \subset \mathrm{End}(A)$ of derivations of A, i.e., linear maps $d : A \to A$ such that

$$d(ab) = d(a) \cdot b + a \cdot d(b).$$

Indeed, consider the action of $GL(A)$ on $\mathrm{Hom}(A \otimes A, A)$. Then $G = G_\mu$ where $\mu : A \otimes A \to A$ is the multiplication map. Also, if $g_t(ab) = g_t(a)g_t(b)$ and $d = \frac{d}{dt}|_{t=0} g_t$ then $d(ab) = d(a) \cdot b + a \cdot d(b)$ and conversely, if d is a derivation then $g_t := \exp(td)$ is an automorphism.

9.2. **The center of G and $\mathfrak{g}$.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $Z = Z(G)$ the center of G, i.e. the set of $z \in G$ such that $zg = gz$ for all $g \in G$. Also let $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ be the set of $x \in \mathfrak{g}$ such that $[x, y] = 0$ for all $y \in \mathfrak{g}$; it is called the **center** of $\mathfrak{g}$.

Proposition 9.8. *If G is connected then Z is a closed (normal, commutative) Lie subgroup of G with Lie algebra $\mathfrak{z}$.*

Proof. Since G is connected, an element $g \in G$ belongs to Z iff it commutes with $\exp(tu)$ for all $u \in \mathfrak{g}$, i.e., iff $\mathrm{Ad}_g(u) = u$. Thus $Z = \mathrm{Ker}(\mathrm{Ad})$, where $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation. Thus by Proposition 4.7, $Z \subset G$ is a closed Lie subgroup with Lie algebra Ker(ad), as claimed. □

Remark 9.9. In general (when G is not necessarily connected), it is easy to show that G/G° acts on $\mathfrak{z}$, and Z is a closed Lie subgroup of G with Lie algebra $\mathfrak{z}^{G/G^\circ}$ (the subspace of invariant vectors).

Definition 9.10. For a connected Lie group G, the group $G/Z(G)$ is called the **adjoint group** of G.

It is clear that $G/Z(G)$ is naturally isomorphic to the image of the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$, which motivates the terminology.

9.3. The statements of the fundamental theorems of Lie theory.

Theorem 9.11. *(First fundamental theorem of Lie theory) For a Lie group G, there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}G$, given by $\mathfrak{h} = \mathrm{Lie}H$.*

Theorem 9.12. *(Second fundamental theorem of Lie theory) If G and K are Lie groups with G simply connected then the map*

$$\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$$

given by $\phi \mapsto \phi_*$ is a bijection.

Theorem 9.13. *(Third fundamental theorem of Lie theory) Any finite dimensional Lie algebra is the Lie algebra of a Lie group.*

These theorems hold for real as well as complex Lie groups. Thus we have

Corollary 9.14. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the assignment $G \mapsto \mathrm{Lie}G$ is an equivalence between the category of simply connected $\mathbb{K}$-Lie groups and the category of finite dimensional $\mathbb{K}$-Lie algebras. Moreover, any connected Lie group K has the form G/Γ where G ‘is simply connected and $\Gamma \subset G$ is a discrete central subgroup.*

Proof. The second fundamental theorem says that the functor $G \mapsto \mathrm{Lie}G$ is fully faithful, and the third fundamental theorem says that it is essentially surjective. Thus it is an equivalence of categories. The last statement follows from Proposition 3.5 (G is the universal covering of K). $\square$

We will discuss proofs of the fundamental theorems of Lie theory in Subsection 10.2. The third theorem is the hardest one, and we will give its complete proof only in Section 49.

9.4. **Complexification of real Lie groups and real forms of complex Lie groups.** Let $\mathfrak{g}$ be a real Lie algebra. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie algebra. We say that $\mathfrak{g}_{\mathbb{C}}$ is the **complexification** of $\mathfrak{g}$, and $\mathfrak{g}$ is a **real form** of $\mathfrak{g}_{\mathbb{C}}$.

Note that two non-isomorphic real Lie algebras can have isomorphic complexifications; in other words, the same complex Lie algebra can have different real forms. For example,

$$\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{C})$$

while for $n > 1$,

$$\mathfrak{u}(n) \not\cong \mathfrak{gl}_n(\mathbb{R}),$$

since in the first algebra any element x with nilpotent $\mathrm{ad}x$ must be zero, while in the second one it must not.

Definition 9.15. Let G be a connected complex Lie group and $K \subset G$ a real Lie subgroup such that Lie$K$ is a real form of Lie$G$ (i.e., the natural map Lie$K \otimes_{\mathbb{R}} \mathbb{C} \to$ Lie$G$ is an isomorphism). Then K is called a **real form** of G.

Example 9.16. Both $U(n)$ and $GL_n(\mathbb{R})$ are real forms of $GL_n(\mathbb{C})$ (but $GL_n(\mathbb{R})$ is not connected). Also $GL_n(\mathbb{R})^\circ \subset GL_n(\mathbb{R})$, the subgroup of matrices with positive determinant, is a connected real form of $GL_n(\mathbb{C})$. So with this definition two different real forms (at least one of which is disconnected) may have the same Lie algebra.[8]

Let K be a simply connected real Lie group, and $\mathfrak{g} =$ Lie$K \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of Lie$K$. By the third fundamental theorem of Lie theory, there exists a unique simply connected complex Lie group G such that Lie$G = \mathfrak{g}$. It is called the **complexification** of K. We have a natural homomoprhism $K \to G$ coming from the homomorphism Lie$K \to$ LieG (from the second fundamental theorem), but it need not be injective; e.g. it is not for K being the universal covering of $SL_2(\mathbb{R})$.

[8]Later, when we discuss real forms of semisimple Lie groups, we will need a more restrictive notion of a real form of a complex group, namely the group of fixed points of an antiholomorphic involution. With this more restrictive definition, the disconnected group $GL_n(\mathbb{R})$ is a real form of $GL_n(\mathbb{C})$, but the connected group $GL_n(\mathbb{R})^\circ$ is not.

Exercise 9.17. (i) Classify complex Lie algebras of dimension at most 3, up to isomorphism.

(ii) Classify real Lie algebras of dimension at most 3.

(iii) Classify connected complex and real Lie groups of dimension at most 3.

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