

10. Proofs of the fundamental theorems of Lie theory

10.1. Distributions and the Frobenius theorem. The proofs of the fundamental theorems of Lie theory are based on the notion of an integrable distribution in differential geometry, and the Frobenius theorem about such distributions.

Definition 10.1. A k -dimensional **distribution** on a manifold X is a rank k subbundle $D \subset TX$.

This means that in every tangent space $T_x X$ we fix a k -dimensional subspace D_x which varies regularly with x . In other words, on some neighborhood $U \subset X$ of every $x \in X$, D is spanned by vector fields $\mathbf{v}_1, \dots, \mathbf{v}_k$ linearly independent at every point of U .

Definition 10.2. A distribution D is **integrable** if every point $x \in X$ has a neighborhood U and local coordinates x_1, \dots, x_n on U such that D is defined at every point of U by the equations $dx_{k+1} = \dots = dx_n = 0$, i.e., it is spanned by vector fields $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, k$.

This is equivalent to saying that every point x of X is contained in an **integral submanifold** for D , i.e., an immersed submanifold $S = S_x \subset X$ such that for any $y \in S$ the tangent space $T_y S \subset T_y X$ coincides with D_y . Namely, S_x is the set of all points of $y \in X$ that can be connected to x by a smooth curve $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) = x, \gamma(1) = y$ and $\gamma'(t) \in D_{\gamma(t)}$ for all $t \in [0, 1]$ (show it!).

For this reason an integrable distribution is also called a **foliation** and the integral submanifolds S_x are called the **sheets of the foliation**. The manifold X falls into a disjoint union of such sheets. But note that the sheets need not be closed (i.e., think of the irrational torus winding!)

Example 10.3. A 1-dimensional distribution is the same thing as a **direction field**. It is always integrable, as follows from the existence theorem for ODE, and its integral submanifolds are called **integral curves**. They are geometric realizations of solutions of the corresponding ODE.

However, for $k \geq 2$ a distribution is not always integrable.

Theorem 10.4. (*The Frobenius theorem*) A distribution D is integrable if and only if for every two vector fields \mathbf{v}, \mathbf{w} contained in D , their commutator $[\mathbf{v}, \mathbf{w}]$ is also contained in D .

Example 10.5. Let $\mathbf{v} = \partial_x$, $\mathbf{w} = x\partial_y + \partial_z$ in \mathbb{R}^3 , and D be the 2-dimensional distribution spanned by \mathbf{v}, \mathbf{w} . Then $[\mathbf{v}, \mathbf{w}] = \partial_y \notin D$. So D is not integrable.

Proof. If D is integrable, a vector field is contained in D iff it is tangent to integral submanifolds of D . But the commutator of two vector fields tangent to a submanifold is itself tangent to this submanifold. This establishes the “only if” part.

It remains to prove the “if” part. The proof is by induction in the rank k of D . The base case $k = 0$ is trivial, so it suffices to establish the inductive step. The question is local, so we may work in a neighborhood U of $P \in X$. Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_k \in \text{Vect}(U)$ is a basis of D in U (on every tangent space). By local existence and uniqueness of solutions of ODE, in some local coordinates $x_1, \dots, x_n = z$, the vector field \mathbf{v}_k equals ∂_z . By subtracting from $\mathbf{v}_i, i < k$ a multiple of \mathbf{v}_k we can make sure that \mathbf{v}_i has no ∂_z -component. Then

$$\mathbf{v}_i = \sum_{j=1}^{n-1} a_{ij}(x_1, \dots, x_{n-1}, z) \partial_{x_j}.$$

Thus we have

$$[\partial_z, \mathbf{v}_i] = \sum_{m=1}^{n-1} b_{im}(x_1, \dots, x_{n-1}, z) \mathbf{v}_m.$$

Thus, setting $A = (a_{ij}(x_1, \dots, x_{n-1}, z)), B = (b_{im}(x_1, \dots, x_{n-1}, z))$, we have

$$\partial_z A = BA.$$

Let A_0 be the solution of this linear ODE with $A_0(x_1, \dots, x_{n-1}, 0) = 1$. Then $A = A_0 C$, where $C = C(x_1, \dots, x_{n-1})$ does not depend on z . So we have a basis of D given by

$$\mathbf{w}_i = \sum_j c_{ij}(x_1, \dots, x_{n-1}) \partial_{x_j}.$$

Thus there is a neighborhood U of P which can be represented as $U = (-a, a) \times U'$, where $\dim U' = n - 1$, so that $D = \mathbb{R} \oplus D'$, where D' is a $k - 1$ -dimensional distribution on U' . It is clear that for any two vector fields \mathbf{v}, \mathbf{w} on U' contained in D' , so is $[\mathbf{v}, \mathbf{w}]$. Hence D' is integrable by the induction assumption. Therefore, so is D , justifying the inductive step. □

10.2. Proofs of the fundamental theorems of Lie theory.

10.2.1. *Proof of Theorem 9.11.* Let G be a Lie group with Lie algebra \mathfrak{g} . Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. We need to show that there is a unique (not necessarily closed) connected Lie subgroup $H \subset G$ with

Lie algebra \mathfrak{h} . The proof of existence of H is based on the Frobenius theorem.

Define the distribution D on G by left-translating $\mathfrak{h} \subset \mathfrak{g} = T_1G$, i.e., $D_g = L_g\mathfrak{h}$. So any vector field contained in D is of the form

$$\mathbf{v} = \sum f_i \mathbf{L}_{a_i},$$

where a_i is a basis of \mathfrak{h} and f_i are regular functions. Now if

$$\mathbf{w} = \sum g_j \mathbf{L}_{a_j}$$

is another such field then

$$[\mathbf{v}, \mathbf{w}] = \sum_{i,j} (f_i \mathbf{L}_{a_i}(g_j) \mathbf{L}_{a_j} - g_j \mathbf{L}_{a_j}(f_i) \mathbf{L}_{a_i} + f_i g_j [\mathbf{L}_{a_i}, \mathbf{L}_{a_j}]).$$

But $[a_i, a_j] = \sum_k c_{ij}^k a_k$, so

$$[\mathbf{L}_{a_i}, \mathbf{L}_{a_j}] = \sum_k c_{ij}^k \mathbf{L}_{a_k}.$$

Thus if \mathbf{v}, \mathbf{w} are contained in D then so is $[\mathbf{v}, \mathbf{w}]$. Hence by the Frobenius theorem, D is integrable.

Now consider the integral (embedded) submanifold H of D going through $1 \in G$. We claim that H is a Lie subgroup of G with Lie algebra \mathfrak{h} . Indeed, it suffices to show that H is a subgroup of G . But this is clear since H is the collection of elements of G of the form

$$g = \exp(a_1) \dots \exp(a_m),$$

where $a_i \in \mathfrak{h}$.

Moreover, H is unique since it has to be generated by the image of the exponential map $\exp : \mathfrak{h} \rightarrow G$.

10.3. Proof of Theorem 9.12. We need to show that the natural map $\text{Hom}(G, K) \rightarrow \text{Hom}(\text{Lie}G, \text{Lie}K)$ is a bijection if G is simply connected.

We know this map is injective so we only need to establish surjectivity. For any morphism $\psi : \text{Lie}G \rightarrow \text{Lie}K$, consider the morphism

$$\theta = (\text{id}, \psi) : \text{Lie}G \rightarrow \text{Lie}(G \times K) = \text{Lie}G \oplus \text{Lie}K$$

The previous proposition implies that there is a connected Lie subgroup $H \subset G \times K$ whose Lie algebra is $\text{Im}\theta$. We have projection homomorphisms $p_1 : H \rightarrow G$, $p_2 : H \rightarrow K$, and $(p_1)_* = \text{id}$, so p_1 is a covering. Since G is simply connected, p_1 is an isomorphism, so we can define $\phi := p_2 \circ p_1^{-1} : G \rightarrow K$, and it is easy to see that $\psi = \phi_*$.

10.4. **Proof of Theorem 9.13.** Finally, let us discuss a proof of Theorem 9.13, stating that any finite dimensional Lie algebra \mathfrak{g} over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} is the Lie algebra of a Lie group. We will deduce it from the following purely algebraic *Ado's theorem*.

Theorem 10.6. *Any finite dimensional Lie algebra over \mathbb{K} is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.*

Ado's theorem in fact holds over any ground field, but it is rather nontrivial and we won't prove it now. A proof can be found, for example, in [J]. But Ado's theorem immediately implies Theorem 9.13. Indeed, using Theorem 9.11, Ado's theorem implies the following even stronger statement:

Theorem 10.7. *Any finite dimensional \mathbb{K} -Lie algebra is the Lie algebra of a Lie subgroup of $\mathfrak{gl}_n(\mathbb{K})$ for some n .*

This implies

Corollary 10.8. *Any simply connected Lie group is the universal covering of a linear Lie group, i.e., of a Lie subgroup of $GL_n(\mathbb{K})$.*

However, it is not true that any Lie group is isomorphic to a Lie subgroup of $GL_n(\mathbb{K})$, see Exercise 11.20.

One can also prove Theorem 9.13 directly and then deduce Ado's theorem as a corollary. We will do this in Sections 49 and 50. We note that Theorem 9.13 will not be used in proofs of other results until that point.

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