12. The universal enveloping algebra of a Lie algebra

12.1. The definition of the universal enveloping algebra. Let $V$ be a vector space over a field $k$. Recall that the tensor algebra of $V$ is the $\mathbb{Z}$-graded associative algebra $TV := \oplus_{n \geq 0} V^\otimes n$ (with $\deg(V^\otimes n) = n$), with multiplication given by $a \cdot b = a \otimes b$ for $a \in V^\otimes m$ and $b \in V^\otimes n$. If $\{x_i\}$ is a basis of $V$ then $TV$ is just the free algebra with generators $x_i$ (i.e., without any relations). Its basis consists of various words in the letters $x_i$.

Let $g$ be a Lie algebra over $k$.

**Definition 12.1.** The universal enveloping algebra of $g$, denoted $U(g)$, is the quotient of $Tg$ by the ideal $I$ generated by the elements $xy - yx - [x, y]$, $x, y \in g$.

Recall that any associative algebra $A$ is also a Lie algebra with operation $[a, b] := ab - ba$. The following proposition follows immediately from the definition of $U(g)$.

**Proposition 12.2.** (i) Let $J \subset Tg$ be an ideal, and $\rho : g \to Tg/J$ the natural linear map. Then $\rho$ is a homomorphism of Lie algebras if and only if $J \supset I$, so that $Tg/J$ is a quotient of $Tg/I = U(g)$. In other words, $U(g)$ is the largest quotient of $Tg$ for which $\rho$ is a homomorphism of Lie algebras.

(ii) Let $A$ be any associative algebra over $k$. Then the map

$$\text{Hom}_{\text{associative}}(U(g), A) \to \text{Hom}_{\text{Lie}}(g, A)$$

given by $\phi \mapsto \phi \circ \rho$ is a bijection.

Part (ii) of this proposition implies that any Lie algebra map $\psi : g \to A$ can be uniquely extended to an associative algebra map $\phi : U(g) \to A$ so that $\psi = \phi \circ \rho$. This is the universal property of $U(g)$ which justifies the term “universal enveloping algebra”.

In particular, it follows that a representation of $g$ on a vector space $V$ is the same thing as an algebra map $U(g) \to \text{End}(V)$ (i.e., a representation of $U(g)$ on $V$). Thus, to understand the representation theory of $g$, it is helpful to understand the structure of $U(g)$; for example, every central element $C \in U(g)$ gives rise to a morphism of representations $V \to V$ (note that this has already come in handy in studying representations of $\mathfrak{sl}_2$).

In terms of the basis $\{x_i\}$ of $g$, we can write the bracket as

$$[x_i, x_j] = \sum_k c^i_{ij} x_k,$$
where \( c_{ij}^k \in k \) are the **structure constants**. Then the algebra \( U(g) \)
can be described as the quotient of the free algebra \( k\langle\{x_i\}\rangle \) by the relations
\[
x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k.
\]

**Example 12.3.** 1. If \( g \) is abelian (i.e., \( c_{ij}^k = 0 \)) then \( U(g) = Sg = k\langle\{x_i\}\rangle \) is the symmetric algebra of \( g \), \( Sg = \oplus_{n \geq 0} S^n g \), which in terms of
the basis is the polynomial algebra in \( x_i \).
2. \( U(sl_2(k)) \) is generated by \( e, f, h \) with defining relations
\[
he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h.
\]
Recall that \( g \) acts on \( Tg \) by derivations via the adjoint action. Moreover, using the Jacobi identity, we have
\[
ad(z(yx - yx - [x, y])) = [z, x]y + x[z, y] - [z, y]x - y[z, x] - [z, [x, y]] = ([z, x]y - y[z, x] - [z, x], y]) + (x[z, y] - [z, y]x - [x, [z, y]])).
\]
Thus \( ad(I) \subset I \), and hence the action of \( g \) on \( Tg \) descends to its action on \( U(g) \) by derivations (also called the adjoint action). It is easy to see that these derivations are in fact inner:
\[
ad(z(a) = za - az
\]
for \( a \in U(g) \) (although this is not so for \( Tg \)). Indeed, it suffices to note that this holds for \( a \in g \) by the definition of \( U(g) \).

Thus we get

**Proposition 12.4.** The center \( Z(U(g)) \) of \( U(g) \) coincides with the subalgebra of invariants \( U(g)^{ad} \).

**Example 12.5.** The Casimir operator \( C = 2fe + \frac{h^2}{2} + h \) which we used to study representations of \( g = sl_2 \) is in fact a central element of \( U(g) \).

12.2. **Graded and filtered algebras.** Recall that a \( \mathbb{Z}_{\geq 0} \)-**filtered** algebra is an algebra \( A \) equipped with a filtration
\[
0 = F_{-1}A \subset F_0A \subset F_1A \subset \ldots \subset F_nA \subset \ldots
\]
such that \( 1 \in F_0A, \cup_{n \geq 0} F_nA = A \) and \( F_iA \cdot F_jA \subset F_{i+j}A \). In particular, if \( A \) is generated by \( \{x_\alpha\} \) then a filtration on \( A \) can be obtained by declaring \( x_\alpha \) to be of degree \( 1 \); i.e., \( F_nA = (F_1A)^n \) is the span of all words in \( x_\alpha \) of degree \( \leq n \).

If \( A = \bigoplus_{i \geq 0} A_i \) is \( \mathbb{Z}_{\geq 0} \)-graded then we can define a filtration on \( A \) by setting \( F_nA := \bigoplus_{i = n}^\infty A_i \); however, not any filtered algebra is obtained in this way, and having a filtration is a weaker condition than having a grading. Still, if \( A \) is a filtered algebra, we can define its **associated graded algebra** \( gr(A) := \bigoplus_{n \geq 0} gr_n(A) \), where \( gr_n(A) := F_nA/F_{n-1}A \).
The multiplication in $\text{gr}(A)$ is given by the “leading terms” of multiplication in $A$: for $a \in F_i A$, $b \in F_j A$, pick their representatives $\tilde{a} \in F_i A$, $\tilde{b} \in F_j A$ and let $ab$ be the projection of $\tilde{a} \tilde{b}$ to $\text{gr}_{i+j}(A)$.

**Proposition 12.6.** If $\text{gr}(A)$ is a domain (has no zero divisors) then so is $A$.

**Exercise 12.7.** Prove Proposition 12.6.

12.3. **The coproduct of $U(g)$**. For a vector space $g$ define the algebra homomorphism $\Delta : Tg \to Tg \otimes Tg$ given for $x \in g \subset Tg$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$ (it exists and is unique since $Tg$ is freely generated by $g$).

**Lemma 12.8.** If $g$ is a Lie algebra then the kernel $I$ of the map $Tg \to U(g)$ satisfies the property $\Delta(I) \subset I \otimes Tg + Tg \otimes I \subset Tg \otimes Tg$. Thus $\Delta$ descends to an algebra homomorphism $U(g) \to U(g) \otimes U(g)$.

**Proof.** For $x, y \in g$ and $a = a(x, y) := xy - yx - [x, y]$ we have $\Delta(a) = a \otimes 1 + 1 \otimes a$. The lemma follows since the ideal $I$ is generated by elements of the form $a(x, y)$.

The homomorphism $\Delta$ is called the **coproduct** (of $Tg$ or $U(g)$).

**Example 12.9.** Let $g = V$ be abelian (a vector space). Then $U(g) = SV$, which for $\dim V < \infty$ can be viewed as the algebra of polynomial functions on $V^*$. Similarly, $SV \otimes SV$ is the algebra of polynomial functions on $V^* \times V^*$. In terms of this identification, we have $\Delta(f)(x, y) = f(x + y)$.