12. **The universal enveloping algebra of a Lie algebra**

12.1. **The definition of the universal enveloping algebra.** Let V be a vector space over a field $\mathbf{k}$. Recall that the **tensor algebra** of V is the $\mathbb{Z}$-graded associative algebra $TV := \oplus_{n\geq 0} V^{\otimes n}$ (with $\deg(V^{\otimes n}) = n$), with multiplication given by $a \cdot b = a \otimes b$ for $a \in V^{\otimes m}$ and $b \in V^{\otimes n}$. If $\{x_i\}$ is a basis of V then TV is just the free algebra with generators x_i (i.e., without any relations). Its basis consists of various words in the letters x_i.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{k}$.

Definition 12.1. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the quotient of $T\mathfrak{g}$ by the ideal I generated by the elements $xy - yx - [x, y]$, $x, y \in \mathfrak{g}$.

Recall that any associative algebra A is also a Lie algebra with operation $[a, b] := ab - ba$. The following proposition follows immediately from the definition of $U(\mathfrak{g})$.

Proposition 12.2. *(i) Let $J \subset T\mathfrak{g}$ be an ideal, and $\rho : \mathfrak{g} \to T\mathfrak{g}/J$ the natural linear map. Then ρ is a homomorphism of Lie algebras if and only if $J \supset I$, so that $T\mathfrak{g}/J$ is a quotient of $T\mathfrak{g}/I = U(\mathfrak{g})$. In other words, $U(\mathfrak{g})$ is the largest quotient of $T\mathfrak{g}$ for which ρ is a homomorphism of Lie algebras.*

(ii) Let A be any associative algebra over $\mathbf{k}$. Then the map

$$\mathrm{Hom}_{\mathrm{associative}}(U(\mathfrak{g}), A) \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A)$$

given by $\phi \mapsto \phi \circ \rho$ is a bijection.

Part (ii) of this proposition implies that any Lie algebra map $\psi : \mathfrak{g} \to A$ can be uniquely extended to an associative algebra map $\phi : U(\mathfrak{g}) \to A$ so that $\psi = \phi \circ \rho$. This is the universal property of $U(\mathfrak{g})$ which justifies the term "universal enveloping algebra".

In particular, it follows that a representation of $\mathfrak{g}$ on a vector space V is the same thing as an algebra map $U(\mathfrak{g}) \to \mathrm{End}(V)$ (i.e., a representation of $U(\mathfrak{g})$ on V). Thus, to understand the representation theory of $\mathfrak{g}$, it is helpful to understand the structure of $U(\mathfrak{g})$; for example, every central element $C \in U(\mathfrak{g})$ gives rise to a morphism of representations $V \to V$ (note that this has already come in handy in studying representations of $\mathfrak{sl}_2$).

In terms of the basis $\{x_i\}$ of $\mathfrak{g}$, we can write the bracket as

$$[x_i, x_j] = \sum_k c_{ij}^k x_k,$$

where $c_{ij}^k \in \mathbf{k}$ are the **structure constants**. Then the algebra $U(\mathfrak{g})$ can be described as the quotient of the free algebra $\mathbf{k}\langle\{x_i\}\rangle$ by the relations

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k.$$

Example 12.3. 1. If $\mathfrak{g}$ is abelian (i.e., $c_{ij}^k = 0$) then $U(\mathfrak{g}) = S\mathfrak{g} = \mathbf{k}[\{x_i\}]$ is the symmetric algebra of $\mathfrak{g}$, $S\mathfrak{g} = \oplus_{n\geq 0} S^n\mathfrak{g}$, which in terms of the basis is the polynomial algebra in x_i.

2. $U(\mathfrak{sl}_2(\mathbf{k}))$ is generated by e, f, h with defining relations

$$he - eh = 2e,\ hf - fh = -2f,\ ef - fe = h.$$

Recall that $\mathfrak{g}$ acts on $T\mathfrak{g}$ by derivations via the adjoint action. Moreover, using the Jacobi identity, we have

$$\mathrm{ad}z(xy - yx - [x,y]) = [z,x]y + x[z,y] - [z,y]x - y[z,x] - [z,[x,y]] =$$
$$([z,x]y - y[z,x] - [[z,x],y]) + (x[z,y] - [z,y]x - [x,[z,y]]).$$

Thus $\mathrm{ad}z(I) \subset I$, and hence the action of $\mathfrak{g}$ on $T\mathfrak{g}$ descends to its action on $U(\mathfrak{g})$ by derivations (also called the adjoint action). It is easy to see that these derivations are in fact inner:

$$\mathrm{ad}z(a) = za - az$$

for $a \in U(\mathfrak{g})$ (although this is not so for $T\mathfrak{g}$). Indeed, it suffices to note that this holds for $a \in \mathfrak{g}$ by the definition of $U(\mathfrak{g})$.

Thus we get

Proposition 12.4. *The center $Z(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ coincides with the subalgebra of invariants $U(\mathfrak{g})^{\mathrm{ad}\mathfrak{g}}$.*

Example 12.5. The Casimir operator $C = 2fe + \frac{h^2}{2} + h$ which we used to study representations of $\mathfrak{g} = \mathfrak{sl}_2$ is in fact a central element of $U(\mathfrak{g})$.

12.2. **Graded and filtered algebras.** Recall that a $\mathbb{Z}_{\geq 0}$-**filtered** algebra is an algebra A equipped with a filtration

$$0 = F_{-1}A \subset F_0A \subset F_1A \subset ... \subset F_nA \subset ...$$

such that $1 \in F_0A$, $\cup_{n\geq 0} F_nA = A$ and $F_iA \cdot F_jA \subset F_{i+j}A$. In particular, if A is generated by $\{x_\alpha\}$ then a filtration on A can be obtained by declaring x_α to be of degree 1; i.e., $F_nA = (F_1A)^n$ is the span of all words in x_α of degree $\leq n$.

If $A = \oplus_{i\geq 0} A_i$ is $\mathbb{Z}_{\geq 0}$-graded then we can define a filtration on A by setting $F_nA := \oplus_{i=0}^n A_i$; however, not any filtered algebra is obtained in this way, and having a filtration is a weaker condition than having a grading. Still, if A is a filtered algebra, we can define its **associated graded algebra** $\mathrm{gr}(A) := \oplus_{n\geq 0} \mathrm{gr}_n(A)$, where $\mathrm{gr}_n(A) := F_nA/F_{n-1}A$.

The multiplication in $\mathrm{gr}(A)$ is given by the "leading terms" of multiplication in A: for $a \in F_iA$, $b \in F_jA$, pick their representatives $\widetilde{a} \in F_iA$, $\widetilde{b} \in F_jA$ and let ab be the projection of $\widetilde{a}\widetilde{b}$ to $\mathrm{gr}_{i+j}(A)$.

Proposition 12.6. *If* $\mathrm{gr}(A)$ *is a domain (has no zero divisors) then so is* A.

Exercise 12.7. Prove Proposition 12.6.

12.3. **The coproduct of $U(\mathfrak{g})$.** For a vector space $\mathfrak{g}$ define the algebra homomorphism $\Delta : T\mathfrak{g} \to T\mathfrak{g} \otimes T\mathfrak{g}$ given for $x \in \mathfrak{g} \subset T\mathfrak{g}$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$ (it exists and is unique since $T\mathfrak{g}$ is freely generated by $\mathfrak{g}$).

Lemma 12.8. *If* $\mathfrak{g}$ *is a Lie algebra then the kernel* I *of the map* $T\mathfrak{g} \to U(\mathfrak{g})$ *satisfies the property* $\Delta(I) \subset I \otimes T\mathfrak{g} + T\mathfrak{g} \otimes I \subset T\mathfrak{g} \otimes T\mathfrak{g}$. *Thus* Δ *descends to an algebra homomorphism* $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

Proof. For $x, y \in \mathfrak{g}$ and $a = a(x, y) := xy - yx - [x, y]$ we have $\Delta(a) = a \otimes 1 + 1 \otimes a$. The lemma follows since the ideal I is generated by elements of the form $a(x, y)$. □

The homomorphism Δ is called the **coproduct** (of $T\mathfrak{g}$ or $U(\mathfrak{g})$).

Example 12.9. Let $\mathfrak{g} = V$ be abelian (a vector space). Then $U(\mathfrak{g}) = SV$, which for $\dim V < \infty$ can be viewed as the algebra of polynomial functions on V^*. Similarly, $SV \otimes SV$ is the algebra of polynomial functions on $V^* \times V^*$. In terms of this identification, we have $\Delta(f)(x, y) = f(x + y)$.

18.745 Lie Groups and Lie Algebras I
Fall 2020