14. **Free Lie algebras, the Baker-Campbell-Hausdorff formula**

14.1. **Primitive elements.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbf{k}$. Let us say that $x \in U(\mathfrak{g})$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$. It is clear that if $x \in \mathfrak{g} \subset U(\mathfrak{g})$ then x is primitive.

Lemma 14.1. *If the ground field* $\mathbf{k}$ *has characteristic zero then every primitive element of* $U(\mathfrak{g})$ *is contained in* $\mathfrak{g}$.

Proof. Let $0 \neq f \in U(\mathfrak{g})$ be a primitive element. Suppose that the filtration degree of f is n. Let $f_0 \in S^n\mathfrak{g}$ be the leading term of f (it is well defined by the PBW Theorem). Then f_0 is primitive in $S\mathfrak{g}$, and in fact in SV for some finite dimensional subspace $V \subset \mathfrak{g}$. So $f_0(x+y) = f_0(x) + f_0(y)$, $x, y \in V^*$. In particular, $2^n f_0(x) = f_0(2x) = 2f_0(x)$, so $2^n - 2 = 0$, which implies that $n = 1$ as $\mathrm{char}(\mathbf{k}) = 0$. Thus $f = c + f_0$ where $f_0 \in \mathfrak{g}$, $c \in \mathbf{k}$ and $c = 0$ since f is primitive. $\square$

Remark 14.2. Note that the assumption of characteristic zero is essential. Indeed, if the characterictic of $\mathbf{k}$ is $p > 0$ and $x \in \mathfrak{g}$ then $x^{p^i} \in U(\mathfrak{g})$ is primitive for all i.

14.2. **Free Lie algebras.** Let V be a vector space over a field $\mathbf{k}$. The **free Lie algebra** $L(V)$ generated by V is the Lie subalgebra of TV generated by V. Note that $L(V)$ is a $\mathbb{Z}_{>0}$-graded Lie algebra: $L(V) = \oplus_{m \geq 1} L_m(V)$, with grading defined by $\deg V = 1$; thus $L_m(V)$ is spanned by commutators of m-tuples of elements of V inside TV.

Example 14.3. The free Lie algebra $FL_2 = L(\mathbf{k}^2)$ in two generators x, y is generated by x, y with $FL_2[1]$ having basis x, y, $FL_2[2]$ having basis $[x, y]$, $FL_2[3]$ having basis $[x, [x, y]]$, $[y, [x, y]]$, etc. Similarly, $FL_3 = L(\mathbf{k}^3)$ is generated by x, y, z with $FL_3[1]$ having basis x, y, z, $FL_3[2]$ having basis $[x, y], [x, z], [y, z]$, $FL_3[3]$ having basis $[x, [x, y]]$, $[y, [x, y]]$, $[y, [y, z]]$, $[z, [y, z]]$, $[x, [x, z]$, $[z, [x, z]]$, $[x, [y, z]]$, $[y, [z, x]]$ (note that $[z, [x, y]]$ expresses in terms of the last two using the Jacobi identity).

The Lie algebra embedding $L(V) \hookrightarrow TV$ gives rise to an associative algebra homomorphism $\psi : U(L(V)) \to TV$.

Proposition 14.4. *(i)* ψ *is an isomorphism, so* $U(L(V)) \cong TV$.

(ii) ψ *preserves the coproduct.*

(iii) (The universal property of free Lie algebras) If $\mathfrak{g}$ *is any Lie algebra over* $\mathbf{k}$ *then restriction to* V *defines an isomorphism*

$$\mathbf{res} : \mathrm{Hom}_{\mathrm{Lie}}(L(V), \mathfrak{g}) \cong \mathrm{Hom}_{\mathbf{k}}(V, \mathfrak{g}).$$

Proof. (i) By definition, $U(L(V))$ is generated by V as an associative algebra, so $U(L(V)) = TV/J$ for some 2-sided ideal J. Moreover, the map $\psi : TV/J \to TV$ restricts to the identity on the space V of generators. Thus $J = 0$ and $\psi = \mathrm{Id}$.

(ii) is clear since the two coproducts agree on generators.

(iii) Let $a : V \to \mathfrak{g}$ be a linear map. Then a can be viewed as a linear map $V \to U(\mathfrak{g})$. So it extends to a map of associative algebras $\widetilde{a} : TV \to U(\mathfrak{g})$ which restricts to a Lie algebra map $\widehat{a} : L(V) \to U(\mathfrak{g})$. Moreover, since $\widehat{a}(V) \subset \mathfrak{g} \subset U(\mathfrak{g})$ and $L(V)$ is generated by V as a Lie algebra, we obtain that $\widehat{a} : L(V) \to \mathfrak{g}$. It is easy to see that the assignment $a \mapsto \widehat{a}$ is inverse to **res**, implying that **res** is an isomorphism. □

Exercise 14.5. Let $\dim V = n$ and $d_m(n) = \dim L_m(V)$. Use the PBW theorem to show that $d_m(n)$ are uniquely determined from the identity

$$\prod_{m=1}^{\infty} (1 - q^m)^{d_m(n)} = 1 - nq.$$

14.3. **The Baker-Campbell-Hausdorff formula.** We have defined the commutator $[x, y]$ on $\mathfrak{g} = \mathrm{Lie}G$ as the quadratic part of $\mu(x, y) = \log(\exp(x)\exp(y))$. So one may wonder if taking higher order terms in the Taylor explansion of $\mu(x, y)$,

$$\mu(x, y) = \sum_{n=1}^{\infty} \frac{\mu_n(x, y)}{n!}$$

would yield new operations on $\mathfrak{g}$. It turns out, however, that all these operations express via the commutator. Namely, we have

Theorem 14.6. *For each $n \geq 1$, $\mu_n(x, y)$ may be written as a $\mathbb{Q}$-Lie polynomial of x, y (i.e., a $\mathbb{Q}$-linear combination of Lie monomials, obtained by taking successive commutators of x, y), which is universal (i.e., independent on G).*

Proof. Let $T\mathbb{C}^2 = \mathbb{C}\langle x, y\rangle$ be the free noncommutative algebra in the letters x, y. The series $X = \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ can be viewed as as an element of the degree completion $\widehat{\mathbb{C}\langle x, y\rangle}$ (i.e., the space of noncommutative formal series in x and y), and similarly for $Y := \exp(y)$. Thus we may define

$$\mu := \log(XY) \in \widehat{\mathbb{C}\langle x, y\rangle},$$

where

$$\log A := -\sum_{n=1}^{\infty} \frac{(1 - A)^n}{n}.$$

Then $\mu = \sum_{n=1}^{\infty} \frac{\mu_n}{n!}$ where $\mu_n \in \mathbb{C}\langle x, y\rangle$ is homogeneous of degree n. These μ_n are the desired universal expressions, and it remains to show that they are Lie polynomials, i.e., can be expressed solely in terms of commutators.

To this end, note that since $\Delta(x) = x \otimes 1 + 1 \otimes x$, the element X is **grouplike**, i.e., $\Delta(X) = X \otimes X$ (where we extend the coproduct to the completion by continuity). The same property is shared by Y and hence by $Z := XY$, i.e., we have $\Delta(Z) = Z \otimes Z$. Thus

$$\begin{aligned}\Delta(\log Z) = \log \Delta(Z) = \log(Z \otimes Z) = \log((Z \otimes 1)(1 \otimes Z)) \\ = \log Z \otimes 1 + 1 \otimes \log Z.\end{aligned}$$

Thus $\mu = \log Z$ is primitive, hence so is μ_n for each n. Thus by Lemma 14.1, $\mu_n \in FL_2 = L(\mathbb{C}^2)$, where $FL_2 \subset \mathbb{C}\langle x, y\rangle$ is the free Lie algebra generated by x, y. This implies the statement. □

Example 14.7.

$$\mu_3(x, y) = \frac{1}{2}([x, [x, y]] + [y, [y, x]]).$$

Thus

$$\mu(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + ...$$

Remark 14.8. E. Dynkin derived an explicit formula for $\mu(x, y)$ making it apparent that it expresses solely in terms of commutators. Several proofs of this formula may be found in the expository paper [Mu].