15. **Solvable and nilpotent Lie algebras, theorems of Lie and Engel**

15.1. **Ideals and commutant.** Let $\mathfrak{g}$ be a Lie algebra. Recall that an ideal in $\mathfrak{g}$ is a subspace $\mathfrak{h}$ such that $[\mathfrak{g},\mathfrak{h}]=\mathfrak{h}$. If $\mathfrak{h}\subset\mathfrak{g}$ is an ideal then $\mathfrak{g}/\mathfrak{h}$ has a natural structure of a Lie algebra. Moreover, if $\phi:\mathfrak{g}_1\to\mathfrak{g}_2$ is a homomoprphism of Lie algebras then $\mathrm{Ker}\phi$ is an ideal in $\mathfrak{g}_1$, $\mathrm{Im}\phi$ is a Lie subalgebra in $\mathfrak{g}_2$, and ϕ induces an isomorphism $\mathfrak{g}_1/\mathrm{Ker}\phi\cong\mathrm{Im}\phi$ (check it!).

Lemma 15.1. *If $I_1,I_2\subset\mathfrak{g}$ are ideals then so are $I_1\cap I_2, I_1+I_2,[I_1,I_2]$.*

Exercise 15.2. Prove Lemma 15.1.

Definition 15.3. The commutant of $\mathfrak{g}$ is the ideal $[\mathfrak{g},\mathfrak{g}]$.

Lemma 15.4. *The quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian; moreover, if $I\subset\mathfrak{g}$ is an ideal such that $\mathfrak{g}/I$ is abelian then $I\supset[\mathfrak{g},\mathfrak{g}]$.*

Exercise 15.5. Prove Lemma 15.4.

Example 15.6. The commutant of $\mathfrak{gl}_n(\mathbf{k})$ is $\mathfrak{sl}_n(\mathbf{k})$ (check it!).

Exercise 15.7. (i) Prove that if G is a connected Lie group with Lie algebra $\mathfrak{g}$ then the group commutant $[G,G]$ (the subgroup of G generated by elements $ghg^{-1}h^{-1}$, $g,h\in G$) is a Lie subgroup of G with Lie algebra $[\mathfrak{g},\mathfrak{g}]$.

(ii) Let $\widetilde{G}=\mathbb{R}\times H$, where H is the **Heisenberg group** of real matrices of the form

$$M(a,b,c):=\begin{pmatrix}1&a&b\\0&1&c\\0&0&1\end{pmatrix},\ a,b,c\in\mathbb{R}.$$

Let $\Gamma\cong\mathbb{Z}^2\subset\widetilde{G}$ be the (closed) central subgroup generated by the pairs $(1,M(0,0,0)=\mathrm{Id})$ and $(\sqrt{2},M(0,0,1))$. Let $G=\widetilde{G}/\Gamma$. Show that $[G,G]$ is not closed in G (although by (i) it is a Lie subgroup).

(iii) Does $[G,G]$ have to be closed in G if G is simply connected? (Consider $\mathrm{Hom}(G,\mathbb{R})$ and apply the second fundamental theorem of Lie theory).

15.2. **Solvable Lie algebras.** For a Lie algebra $\mathfrak{g}$ define its **derived series** recursively by the formulas $D^0(\mathfrak{g})=\mathfrak{g}$, $D^{n+1}(\mathfrak{g})=[D^n(\mathfrak{g}),D^n(\mathfrak{g})]$. This is a descending sequence of ideals in $\mathfrak{g}$.

Definition 15.8. A Lie algebra $\mathfrak{g}$ is said to be **solvable** if $D^n(\mathfrak{g})=0$ for some n.

Proposition 15.9. *The following conditions on* $\mathfrak{g}$ *are equivalent:*

(i) $\mathfrak{g}$ *is solvable;*

(ii) There exists a sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset ... \supset \mathfrak{g}_m = 0$ *such that* $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *is abelian.*

Proof. It is clear that (i) implies (ii), since we can take $\mathfrak{g}_i = D^i\mathfrak{g}$. Conversely, by induction we see that $D^i\mathfrak{g} \subset \mathfrak{g}_i$, as desired. □

Proposition 15.10. *(i) Any Lie subalgebra or quotient of a solvable Lie algebra is solvable.*

(ii) If $I \subset \mathfrak{g}$ *is an ideal and* $I, \mathfrak{g}/I$ *are solvable then* $\mathfrak{g}$ *is solvable.*

Exercise 15.11. Prove Proposition 15.10.

15.3. **Nilpotent Lie algebras.** For a Lie algebra $\mathfrak{g}$ define its **lower central series** recursively by the formulas $D_0(\mathfrak{g}) = \mathfrak{g}$, $D_{n+1}(\mathfrak{g}) = [\mathfrak{g}, D_n(\mathfrak{g})]$. This is a descending sequence of ideals in $\mathfrak{g}$.

Definition 15.12. A Lie algebra $\mathfrak{g}$ is said to be **nilpotent** if $D_n(\mathfrak{g}) = 0$ for some n.

Proposition 15.13. *The following conditions on* $\mathfrak{g}$ *are equivalent:*

(i) $\mathfrak{g}$ *is nilpotent;*

(ii) There exists a sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset ... \supset \mathfrak{g}_m = 0$ *such that* $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$.

Proof. It is clear that (i) implies (ii), since we can take $\mathfrak{g}_i = D_i\mathfrak{g}$. Conversely, by induction we see that $D_i\mathfrak{g} \subset \mathfrak{g}_i$, as desired. □

Remark 15.14. Any nilpotent Lie algebra is solvable since $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ implies $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$, hence $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

Proposition 15.15. *Any Lie subalgebra or quotient of a nilpotent Lie algebra is nilpotent.*

Exercise 15.16. Prove Proposition 15.15.

Example 15.17. (i) The Lie algebra of upper triangular matrices of size n is solvable, but it is not nilpotent for $n \geq 2$.

(ii) The Lie algebra of strictly upper triangular matrices is nilpotent.

(iii) The Lie algebra of all matrices of size $n \geq 2$ is not solvable.

15.4. **Lie's theorem.** One of the main technical tools of the structure theory of finite dimensional Lie algebras is **Lie's theorem** for solvable Lie algebras. Before stating and proving this theorem, we will prove the following auxiliary lemma, which will be used several times.

Lemma 15.18. *Let $\mathfrak{g} = \mathbf{k}x \oplus \mathfrak{h}$ be a Lie algebra over a field $\mathbf{k}$ in which $\mathfrak{h}$ is an ideal. Let V be a finite dimensional $\mathfrak{g}$-module and $v \in V$ a common eigenvector of $\mathfrak{h}$:*

$$av = \lambda(a)v, \ a \in \mathfrak{h}$$

where $\lambda : \mathfrak{h} \to \mathbf{k}$ is a character. Then:

(i) $W := \mathbf{k}[x]v$ is a $\mathfrak{g}$-submodule on V on which $a - \lambda(a)$ is nilpotent for all $a \in \mathfrak{h}$.

(ii) If in addition λ vanishes on $[\mathfrak{g}, \mathfrak{g}]$ (i.e., $\lambda([a, x]) = 0$ for all $a \in \mathfrak{h}$) then every $a \in \mathfrak{h}$ acts on W by the scalar $\lambda(a)$. Thus the common eigenspace $V_\lambda \subset V$ of $\mathfrak{h}$ is a $\mathfrak{g}$-submodule.

(iii) The assumption (hence the conclusion) of (ii) always holds if $\mathrm{char}(\mathbf{k}) = 0$.

Proof. (i) For $a \in \mathfrak{h}$ we have

$$ax^n v = xax^{n-1}v + [a, x]x^{n-1}v. \tag{15.1}$$

Therefore, it follows by induction in n that $ax^n v$ is a linear combination of $v, xv, ..., x^n v$, hence $W \subset V$ is a submodule.

Let n be the smallest integer such that $x^n v$ is a linear combination of $x^i v$ with $i < n$. Then $v_i := x^{i-1}v$ for $i = 1, ..., n$ is a basis of W and $\dim W = n$. It follows from (15.1) that the element a acts in this basis by an upper triangular matrix with all diagonal entries equal $\lambda(a)$, as claimed.

(ii) It follows from (15.1) by induction in n that for every $a \in \mathfrak{h}$, $ax^n v = \lambda(a)x^n v$, as desired.

(iii) By (i), $\mathrm{Tr}(a|_W) = n\lambda(a)$ for all $a \in \mathfrak{h}$. On the other hand, if $a \in [\mathfrak{g}, \mathfrak{g}]$ then $\mathrm{Tr}(a|_W) = 0$, thus $n\lambda(a) = 0$ in $\mathbf{k}$. Since $\mathrm{char}(\mathbf{k}) = 0$, this implies that $\lambda(a) = 0$. ☐

Theorem 15.19. *(Lie's theorem) Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, and $\mathfrak{g}$ a finite dimensional solvable Lie algebra over $\mathbf{k}$. Then any irreducible finite dimensional representation of $\mathfrak{g}$ is 1-dimensional.*

Proof. Let V be a finite dimensional representation of $\mathfrak{g}$. It suffices to show that V contains a common eigenvector of $\mathfrak{g}$. The proof is by induction in $\dim \mathfrak{g}$. The base is trivial so let us justify the induction step. Since $\mathfrak{g}$ is solvable, $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$, so fix a subspace $\mathfrak{h} \subset \mathfrak{g}$ of codimension 1 containing $[\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, hence solvable. Thus by the induction assumption, there is a nonzero common eigenvector $v \in V$ for $\mathfrak{h}$, i.e., there is a linear functional $\lambda : \mathfrak{h} \to \mathbf{k}$ such that $av = \lambda(a)v$ for all $a \in \mathfrak{h}$.

Let $x \in \mathfrak{g}$ be an element not belonging to $\mathfrak{h}$ and W be the subspace of V spanned by $v, xv, x^2v, ...$ By Lemma 15.18(i), W is a $\mathfrak{g}$-submodule of V and $a - \lambda(a)$ is nilpotent on W. Thus by Lemma 15.18(ii),(iii) every $a \in \mathfrak{h}$ acts on W by $\lambda(a)$, in particular $[\mathfrak{g}, \mathfrak{g}]$ acts by zero. Hence W is a representation of the abelian Lie algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Now the statement follows since every finite dimensional representation of an abelian Lie algebra has a common eigenvector. □

Remark 15.20. Lemma 15.18(iii) and Lie's theorem do not hold in characteristic $p > 0$. Indeed, let $\mathfrak{g}$ be the Lie algebra with basis x, y and $[x, y] = y$, and let V be the space with basis $v_0, ..., v_{p-1}$ and action of $\mathfrak{g}$ given by

$$xv_i = iv_i, \ yv_i = v_{i+1},$$

where $i + 1$ is taken modulo p. It is easy to see that V is irreducible.

Here is another formulation of Lie's theorem:

Corollary 15.21. *Every finite dimensional representation V of a finite dimensional solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbf{k}$ of characteristic zero has a basis in which all elements of $\mathfrak{g}$ act by upper triangular matrices. In other words, there is a sequence of subrepresentations $0 = V_0 \subset V_1 \subset ... \subset V_n = V$ such that $\dim(V_{k+1}/V_k) = 1$.*

In the case $\dim \mathfrak{g} = 1$, this recovers the well known theorem in linear algebra that any linear operator on a finite dimensional $\mathbf{k}$-vector space is upper triangular in some basis (which is actually true in any characteristic).

Proof. The proof is by induction in $\dim V$ (where the base is obvious). By Lie's theorem, there is a common eigenvector $v_0 \in V$ for $\mathfrak{g}$. Let $V' := V/\mathbf{k}v_0$. Then by the induction assumption V' has a basis $v_1', ..., v_n'$ in which $\mathfrak{g}$ acts by upper triangular matrices. Let $v_1, ..., v_n$ be any lifts of $v_1', ..., v_n'$ to V. Then $v_0, v_1, ..., v_n$ is a basis of V in which $\mathfrak{g}$ acts by upper triangular matrices. □

Corollary 15.22. *Over an algebraically closed field of characteristic zero, the following hold.*

(i) A solvable finite dimensional Lie algebra $\mathfrak{g}$ admits a sequence of ideals $0 = I_0 \subset I_1 \subset ... \subset I_n = \mathfrak{g}$ such that $\dim(I_{k+1}/I_k) = 1$.

(ii) A finite dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. (i) Apply Corollary 15.21 to the adjoint representation of $\mathfrak{g}$.

(ii) If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent then it is solvable and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, so $\mathfrak{g}$ is solvable. Conversely, if $\mathfrak{g}$ is solvable then by Corollary 15.21 elements of $[\mathfrak{g}, \mathfrak{g}]$ act on $\mathfrak{g}$, hence on $[\mathfrak{g}, \mathfrak{g}]$ by strictly upper triangular matrices, which implies the statement. □

Example 15.23. Let $\mathfrak{g}, V$ be as in Remark 15.20 and $\mathfrak{h} = \mathfrak{g} \ltimes V$ be the semidirect product, i.e. $\mathfrak{h} = \mathfrak{g} \oplus V$ as a space with

$$[(g_1, v_1), (g_2, v_2)] = ([g_1, g_2], g_1 v_2 - g_2 v_1).$$

Then $\mathfrak{h}$ is a counterexample to Corollary 15.22 both (i) and (ii) in characteristic $p > 0$.

15.5. **Engel's theorem.** Another key tool of the structure theory of finite dimensional Lie algebras is **Engel's theorem**. Before stating and proving this theorem, we prove an auxiliary result.

Theorem 15.24. *Let $V \neq 0$ be a finite dimensional vector space over any field $\mathbf{k}$, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra consisting of nilpotent operators. Then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$.*

Proof. The proof is by induction on the dimension of $\mathfrak{g}$. The base case $\mathfrak{g} = 0$ is trivial and we assume the dimension of $\mathfrak{g}$ is positive.

First we find an ideal $\mathfrak{h}$ of codimension one in $\mathfrak{g}$. Let $\mathfrak{h}$ be a maximal (proper) subalgebra of $\mathfrak{g}$, which exists by finite-dimensionality of $\mathfrak{g}$. We claim that $\mathfrak{h} \subset \mathfrak{g}$ is an ideal and has codimension one.

Indeed, for each $a \in \mathfrak{h}$, the operator $\mathrm{ad}a$ induces a linear map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$, and this induced map is nilpotent (in fact, $\mathrm{ad}a : \mathfrak{g} \to \mathfrak{g}$ is nilpotent). Thus, by the inductive hypothesis, there exists a nonzero element $\bar{x}$ in $\mathfrak{g}/\mathfrak{h}$ such that $\mathrm{ad}a \cdot \bar{x} = 0$ for each $a \in \mathfrak{h}$. Let x be a lift of $\bar{x}$ to $\mathfrak{g}$. Then $[a, x] \in \mathfrak{h}$ for all $a \in \mathfrak{g}$. Let $\mathfrak{h}'$ be the span of $\mathfrak{h}$ and x. Then $\mathfrak{h}' \subset \mathfrak{g}$ is a Lie subalgebra in which $\mathfrak{h}$ is an ideal. Hence, by maximality, $\mathfrak{h}' = \mathfrak{g}$. This proves the claim.

Now let $W = V^{\mathfrak{h}} \subset V$. By the inductive hypothesis, $W \neq 0$. Also by Lemma 15.18(ii) (with $\lambda = 0$), W is a $\mathfrak{g}$-subrepresentation of V.

Now take $w \neq 0$ in W. Let k be the smallest positive integer such that $x^k w = 0$; it exists since x acts nilpotently on V. Let $v = x^{k-1} w \in W$. Then $v \neq 0$ but $\mathfrak{h}v = xv = 0$, so $\mathfrak{g}v = 0$, as desired. □

Definition 15.25. An element $x \in \mathfrak{g}$ is said to be **nilpotent** if the operator $\mathrm{ad}x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.

Corollary 15.26. *(Engel's theorem) A finite dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if every element $x \in \mathfrak{g}$ is nilpotent.*

Proof. The "only if" direction is easy. To prove the "if" direction, note that by Theorem 15.24, in some basis v_i of $\mathfrak{g}$ all elements $\mathrm{ad}x$ act by strictly upper triangular matrices. Let I_m be the subspace of $\mathfrak{g}$ spanned by the vectors $v_1, ..., v_m$. Then $I_m \subset I_{m+1}$ and $[\mathfrak{g}, I_{m+1}] \subset I_m$, hence $\mathfrak{g}$ is nilpotent. $\square$