15. Solvable and nilpotent Lie algebras, theorems of Lie and Engel

15.1. Ideals and commutant. Let \( \mathfrak{g} \) be a Lie algebra. Recall that an ideal in \( \mathfrak{g} \) is a subspace \( \mathfrak{h} \) such that \( [\mathfrak{g}, \mathfrak{h}] = \mathfrak{h} \). If \( \mathfrak{h} \subset \mathfrak{g} \) is an ideal then \( \mathfrak{g}/\mathfrak{h} \) has a natural structure of a Lie algebra. Moreover, if \( \phi : \mathfrak{g}_1 \to \mathfrak{g}_2 \) is a homomorphism of Lie algebras then \( \ker \phi \) is an ideal in \( \mathfrak{g}_1 \), \( \text{Im} \phi \) is a Lie subalgebra in \( \mathfrak{g}_2 \), and \( \phi \) induces an isomorphism \( \mathfrak{g}_1/\ker \phi \cong \text{Im} \phi \) (check it!).

Lemma 15.1. If \( I_1, I_2 \subset \mathfrak{g} \) are ideals then so are \( I_1 \cap I_2 \), \( I_1 + I_2 \), \( [I_1, I_2] \).

Exercise 15.2. Prove Lemma 15.1.

Definition 15.3. The commutant of \( \mathfrak{g} \) is the ideal \( [\mathfrak{g}, \mathfrak{g}] \).

Lemma 15.4. The quotient \( \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \) is abelian; moreover, if \( I \subset \mathfrak{g} \) is an ideal such that \( \mathfrak{g}/I \) is abelian then \( I \supset [\mathfrak{g}, \mathfrak{g}] \).

Exercise 15.5. Prove Lemma 15.4.

Example 15.6. The commutant of \( \mathfrak{gl}_n(k) \) is \( \mathfrak{sl}_n(k) \) (check it!).

Exercise 15.7. (i) Prove that if \( G \) is a connected Lie group with Lie algebra \( \mathfrak{g} \) then the group commutant \( [G, G] \) (the subgroup of \( G \) generated by elements \( ghg^{-1}h^{-1} \), \( g, h \in G \)) is a Lie subgroup of \( G \) with Lie algebra \( [\mathfrak{g}, \mathfrak{g}] \).

(ii) Let \( \tilde{G} = \mathbb{R} \times H \), where \( H \) is the Heisenberg group of real matrices of the form

\[
M(a, b, c) := \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{R}.
\]

Let \( \Gamma \cong \mathbb{Z}^2 \subset \tilde{G} \) be the (closed) central subgroup generated by the pairs \((1, M(0, 0, 0) = \text{Id})\) and \((\sqrt{2}, M(0, 0, 1))\). Let \( G = \tilde{G}/\Gamma \). Show that \( [G, G] \) is not closed in \( G \) (although by (i) it is a Lie subgroup).

(iii) Does \( [G, G] \) have to be closed in \( G \) if \( G \) is simply connected? (Consider \( \text{Hom}(G, \mathbb{R}) \) and apply the second fundamental theorem of Lie theory).

15.2. Solvable Lie algebras. For a Lie algebra \( \mathfrak{g} \) define its derived series recursively by the formulas \( D^0(\mathfrak{g}) = \mathfrak{g}, D^{n+1}(\mathfrak{g}) = [D^n(\mathfrak{g}), D^n(\mathfrak{g})] \).

This is a descending sequence of ideals in \( \mathfrak{g} \).

Definition 15.8. A Lie algebra \( \mathfrak{g} \) is said to be solvable if \( D^n(\mathfrak{g}) = 0 \) for some \( n \).
Proposition 15.9. The following conditions on \( g \) are equivalent:

(i) \( g \) is solvable;

(ii) There exists a sequence of ideals \( g = g_0 \supset g_1 \supset \ldots \supset g_m = 0 \) such that \( g_i/g_{i+1} \) is abelian.

Proof. It is clear that (i) implies (ii), since we can take \( g_i = D^i g \). Conversely, by induction we see that \( D^i g \subset g_i \), as desired. \( \square \)

Proposition 15.10. (i) Any Lie subalgebra or quotient of a solvable Lie algebra is solvable.

(ii) If \( I \subset g \) is an ideal and \( I, g/I \) are solvable then \( g \) is solvable.

Exercise 15.11. Prove Proposition 15.10.

15.3. Nilpotent Lie algebras. For a Lie algebra \( g \) define its lower central series recursively by the formulas \( D^0(g) = g, \ D^{n+1}(g) = [g, D^n(g)] \). This is a descending sequence of ideals in \( g \).

Definition 15.12. A Lie algebra \( g \) is said to be nilpotent if \( D^n(g) = 0 \) for some \( n \).

Proposition 15.13. The following conditions on \( g \) are equivalent:

(i) \( g \) is nilpotent;

(ii) There exists a sequence of ideals \( g = g_0 \supset g_1 \supset \ldots \supset g_m = 0 \) such that \( [g, g_i] \subset g_{i+1} \).

Proof. It is clear that (i) implies (ii), since we can take \( g_i = D_i g \). Conversely, by induction we see that \( D_i g \subset g_i \), as desired. \( \square \)

Remark 15.14. Any nilpotent Lie algebra is solvable since \( [g, g_i] \subset g_{i+1} \) implies \( [g, g_i] \subset g_{i+1} \), hence \( g_i/g_{i+1} \) is abelian.

Proposition 15.15. Any Lie subalgebra or quotient of a nilpotent Lie algebra is nilpotent.

Exercise 15.16. Prove Proposition 15.15.

Example 15.17. (i) The Lie algebra of upper triangular matrices of size \( n \) is solvable, but it is not nilpotent for \( n \geq 2 \).

(ii) The Lie algebra of strictly upper triangular matrices is nilpotent.

(iii) The Lie algebra of all matrices of size \( n \geq 2 \) is not solvable.

15.4. Lie’s theorem. One of the main technical tools of the structure theory of finite dimensional Lie algebras is Lie’s theorem for solvable Lie algebras. Before stating and proving this theorem, we will prove the following auxiliary lemma, which will be used several times.
Lemma 15.18. Let $g = kx \oplus h$ be a Lie algebra over a field $k$ in which $h$ is an ideal. Let $V$ be a finite dimensional $g$-module and $v \in V$ a common eigenvector of $h$:

$$av = \lambda(a)v, \ a \in h$$

where $\lambda : h \to k$ is a character. Then:

(i) $W := k[x]v$ is a $g$-submodule on $V$ on which $a - \lambda(a)$ is nilpotent for all $a \in h$.

(ii) If in addition $\lambda$ vanishes on $[g, g]$ (i.e., $\lambda([a, x]) = 0$ for all $a \in h$) then every $a \in h$ acts on $W$ by the scalar $\lambda(a)$. Thus the common eigenspace $V_\lambda \subset V$ of $h$ is a $g$-submodule.

(iii) The assumption (hence the conclusion) of (ii) always holds if $\text{char} (k) = 0$.

Proof. (i) For $a \in h$ we have

$$ax^n v = xax^{n-1}v + [a, x]x^{n-1}v. \tag{15.1}$$

Therefore, it follows by induction in $n$ that $ax^n v$ is a linear combination of $v, xv, ..., x^n v$, hence $W \subset V$ is a submodule.

Let $n$ be the smallest integer such that $x^n v$ is a linear combination of $x^i v$ with $i < n$. Then $v_i := x^{i-1}v$ for $i = 1, ..., n$ is a basis of $W$ and $\dim W = n$. It follows from (15.1) that the element $a$ acts in this basis by an upper triangular matrix with all diagonal entries equal $\lambda(a)$, as claimed.

(ii) It follows from (15.1) by induction in $n$ that for every $a \in h$, $ax^n v = \lambda(a)x^n v$, as desired.

(iii) By (i), $\text{Tr}(a|_W) = n\lambda(a)$ for all $a \in h$. On the other hand, if $a \in [g, g]$ then $\text{Tr}(a|_W) = 0$, thus $n\lambda(a) = 0$ in $k$. Since $\text{char}(k) = 0$, this implies that $\lambda(a) = 0$. □

Theorem 15.19. (Lie’s theorem) Let $k$ be an algebraically closed field of characteristic zero, and $g$ a finite dimensional solvable Lie algebra over $k$. Then any irreducible finite dimensional representation of $g$ is 1-dimensional.

Proof. Let $V$ be a finite dimensional representation of $g$. It suffices to show that $V$ contains a common eigenvector of $g$. The proof is by induction in $\dim g$. The base is trivial so let us justify the induction step. Since $g$ is solvable, $g \neq [g, g]$, so fix a subspace $h \subset g$ of codimension 1 containing $[g, g]$. Since $g/[g, g]$ is abelian, $h$ is an ideal in $g$, hence solvable. Thus by the induction assumption, there is a nonzero common eigenvector $v \in V$ for $h$, i.e., there is a linear functional $\lambda : h \to k$ such that $av = \lambda(a)v$ for all $a \in h$. 82
Let \( x \in \mathfrak{g} \) be an element not belonging to \( \mathfrak{h} \) and \( W \) be the subspace of \( V \) spanned by \( v, xv, x^2v, \ldots \). By Lemma 15.18(i), \( W \) is a \( \mathfrak{g} \)-submodule of \( V \) and \( a - \lambda(a) \) is nilpotent on \( W \). Thus by Lemma 15.18(ii),(iii) every \( a \in \mathfrak{h} \) acts on \( W \) by \( \lambda(a) \), in particular \( [\mathfrak{g}, \mathfrak{g}] \) acts by zero. Hence \( W \) is a representation of the abelian Lie algebra \( \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \). Now the statement follows since every finite dimensional representation of an abelian Lie algebra has a common eigenvector.

**Remark 15.20.** Lemma 15.18(iii) and Lie’s theorem do not hold in characteristic \( p > 0 \). Indeed, let \( \mathfrak{g} \) be the Lie algebra with basis \( x, y \) and \( [x, y] = y \), and let \( V \) be the space with basis \( v_0, \ldots, v_{p-1} \) and action of \( \mathfrak{g} \) given by

\[
xv_i = iv_i, \quad yv_i = v_{i+1},
\]

where \( i + 1 \) is taken modulo \( p \). It is easy to see that \( V \) is irreducible.

Here is another formulation of Lie’s theorem:

**Corollary 15.21.** Every finite dimensional representation \( V \) of a finite dimensional solvable Lie algebra \( \mathfrak{g} \) over an algebraically closed field \( k \) of characteristic zero has a basis in which all elements of \( \mathfrak{g} \) act by upper triangular matrices. In other words, there is a sequence of subrepresentations \( 0 = V_0 \subset V_1 \subset \ldots \subset V_n = V \) such that \( \dim(V_{k+1}/V_k) = 1 \).

In the case \( \dim \mathfrak{g} = 1 \), this recovers the well known theorem in linear algebra that any linear operator on a finite dimensional \( k \)-vector space is upper triangular in some basis (which is actually true in any characteristic).

**Proof.** The proof is by induction in \( \dim V \) (where the base is obvious). By Lie’s theorem, there is a common eigenvector \( v_0 \in V \) for \( \mathfrak{g} \). Let \( V' := V/kv_0 \). Then by the induction assumption \( V' \) has a basis \( v'_1, \ldots, v'_n \) in which \( \mathfrak{g} \) acts by upper triangular matrices. Let \( v_1, \ldots, v_n \) be any lifts of \( v'_1, \ldots, v'_n \) to \( V \). Then \( v_0, v_1, \ldots, v_n \) is a basis of \( V \) in which \( \mathfrak{g} \) acts by upper triangular matrices.

**Corollary 15.22.** Over an algebraically closed field of characteristic zero, the following hold.

(i) A solvable finite dimensional Lie algebra \( \mathfrak{g} \) admits a sequence of ideals \( 0 = I_0 \subset I_1 \subset \ldots \subset I_n = \mathfrak{g} \) such that \( \dim(I_{k+1}/I_k) = 1 \).

(ii) A finite dimensional Lie algebra \( \mathfrak{g} \) is solvable if and only if \( [\mathfrak{g}, \mathfrak{g}] \) is nilpotent.

**Proof.** (i) Apply Corollary 15.21 to the adjoint representation of \( \mathfrak{g} \).

(ii) If \( [\mathfrak{g}, \mathfrak{g}] \) is nilpotent then it is solvable and \( \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \) is abelian, so \( \mathfrak{g} \) is solvable. Conversely, if \( \mathfrak{g} \) is solvable then by Corollary 15.21 elements
of \([g, g]\) act on \(g\), hence on \([g, g]\) by strictly upper triangular matrices, which implies the statement. \(\square\)

**Example 15.23.** Let \(g, V\) be as in Remark 15.20 and \(h = g \ltimes V\) be the semidirect product, i.e. \(h = g \oplus V\) as a space with
\[
[(g_1, v_1), (g_2, v_2)] = ([g_1, g_2], g_1 v_2 - g_2 v_1).
\]
Then \(h\) is a counterexample to Corollary 15.22 both (i) and (ii) in characteristic \(p > 0\).

15.5. **Engel’s theorem.** Another key tool of the structure theory of finite dimensional Lie algebras is **Engel’s theorem.** Before stating and proving this theorem, we prove an auxiliary result.

**Theorem 15.24.** Let \(V \neq 0\) be a finite dimensional vector space over any field \(k\), and \(g \subset \mathfrak{gl}(V)\) be a Lie algebra consisting of nilpotent operators. Then there exists a nonzero vector \(v \in V\) such that \(gv = 0\).

**Proof.** The proof is by induction on the dimension of \(g\). The base case \(g = 0\) is trivial and we assume the dimension of \(g\) is positive.

First we find an ideal \(h\) of codimension one in \(g\). Let \(h\) be a maximal (proper) subalgebra of \(g\), which exists by finite-dimensionality of \(g\). We claim that \(h \subset g\) is an ideal and has codimension one.

Indeed, for each \(a \in h\), the operator \(\text{ad}a\) induces a linear map \(g/h \to g/h\), and this induced map is nilpotent (in fact, \(\text{ad}a : g \to g\) is nilpotent). Thus, by the inductive hypothesis, there exists a nonzero element \(\bar{x}\) in \(g/h\) such that \(\text{ad}a \cdot \bar{x} = 0\) for each \(a \in h\). Let \(x\) be a lift of \(\bar{x}\) to \(g\). Then \([a, x] \in h\) for all \(a \in g\). Let \(h'\) be the span of \(h\) and \(x\). Then \(h' \subset g\) is a Lie subalgebra in which \(h\) is an ideal. Hence, by maximality, \(h' = g\). This proves the claim.

Now let \(W = V^h \subset V\). By the inductive hypothesis, \(W \neq 0\). Also by Lemma 15.18(ii) (with \(\lambda = 0\)), \(W\) is a \(g\)-subrepresentation of \(V\).

Now take \(w \neq 0\) in \(W\). Let \(k\) be the smallest positive integer such that \(x^k w = 0\); it exists since \(x\) acts nilpotently on \(V\). Let \(v = x^{k-1} w \in W\). Then \(v \neq 0\) but \(hv = xv = 0\), so \(gv = 0\), as desired. \(\square\)

**Definition 15.25.** An element \(x \in g\) is said to be **nilpotent** if the operator \(\text{ad}x : g \to g\) is nilpotent.

**Corollary 15.26.** (Engel’s theorem) A finite dimensional Lie algebra \(g\) is nilpotent if and only if every element \(x \in g\) is nilpotent.

**Proof.** The “only if” direction is easy. To prove the “if” direction, note that by Theorem 15.24 in some basis \(v_i\) of \(g\) all elements \(\text{ad}x\) act by strictly upper triangular matrices. Let \(I_m\) be the subspace of \(g\) spanned
by the vectors $v_1, \ldots, v_m$. Then $I_m \subset I_{m+1}$ and $[g, I_{m+1}] \subset I_m$, hence $g$

is nilpotent. □