16. **Semisimple and reductive Lie algebras, the Cartan criteria**

16.1. **Semisimple and reductive Lie algebras, the radical.**

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbf{k}$.

Proposition 16.1. *$\mathfrak{g}$ contains the largest solvable ideal which contains all solvable ideals of $\mathfrak{g}$.*

Definition 16.2. This ideal is called **the radical** of $\mathfrak{g}$ and denoted $\mathrm{rad}(\mathfrak{g})$.

Proof. Let I, J be solvable ideals of $\mathfrak{g}$. Then $I + J \subset \mathfrak{g}$ is an ideal, and $(I + J)/I = J/(I \cap J)$ is solvable, so $I + J$ is solvable. Thus the sum of finitely many solvable ideals is solvable. Hence the sum of all solvable ideals in $\mathfrak{g}$ is a solvable ideal, as desired. □

Definition 16.3. (i) $\mathfrak{g}$ is called **semisimple** if $\mathrm{rad}(\mathfrak{g}) = 0$, i.e., $\mathfrak{g}$ does not contain nonzero solvable ideals.

(ii) A non-abelian $\mathfrak{g}$ is called **simple** if it contains no ideals other than $0, \mathfrak{g}$. In other words, a non-abelian $\mathfrak{g}$ is simple if its adjoint representation is irreducible (=simple).

Thus if $\mathfrak{g}$ is both solvable and semisimple then $\mathfrak{g} = 0$.

Proposition 16.4. *(i) We have $\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) = \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h})$. In particular, the direct sum of semisimple Lie algebras is semisimple.*

(ii) A simple Lie algebra is semisimple. Thus a direct sum of simple Lie algebras is semisimple.

Proof. (i) The images of $\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h})$ in $\mathfrak{g}$ and in $\mathfrak{h}$ are solvable, hence contained in $\mathrm{rad}(\mathfrak{g})$, respectively $\mathrm{rad}(\mathfrak{h})$. Thus

$$\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) \subset \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h}).$$

But $\mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h})$ is a solvable ideal in $\mathfrak{g} \oplus \mathfrak{h}$, so

$$\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) = \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h}).$$

(ii) The only nonzero ideal in $\mathfrak{g}$ is $\mathfrak{g}$, and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ since $\mathfrak{g}$ is not abelian. Hence $\mathfrak{g}$ is not solvable. Thus $\mathfrak{g}$ is semisimple. □

Example 16.5. The Lie algebra $\mathfrak{sl}_2(\mathbf{k})$ is simple if $\mathrm{char}(\mathbf{k}) \neq 2$. Likewise, $\mathfrak{so}_3(\mathbf{k})$ is simple.

Theorem 16.6. *(weak Levi decomposition) The Lie algebra $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. Thus any $\mathfrak{g}$ can be included in an exact sequence*

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}} \to 0,$$

where $\mathrm{rad}(\mathfrak{g})$ *is a solvable ideal and* $\mathfrak{g}_{\mathrm{ss}}$ *is semisimple. Moreover, if* $\mathfrak{h} \subset \mathfrak{g}$ *is a solvable ideal such that* $\mathfrak{g}/\mathfrak{h}$ *is semisimple then* $\mathfrak{h} = \mathrm{rad}(\mathfrak{g})$.

Proof. Let $I \subset \mathfrak{g}_{\mathrm{ss}}$ be a solvable ideal, and let $\widetilde{I}$ be its preimage in $\mathfrak{g}$. Then $\widetilde{I}$ is a solvable ideal in $\mathfrak{g}$. Thus $\widetilde{I} = \mathrm{rad}(\mathfrak{g})$ and $I = 0$. □

In fact, in characteristic zero there is a stronger statement, which says that the extension in Theorem 16.6 splits. Namely, given a Lie algebra $\mathfrak{h}$ and another Lie algebra $\mathfrak{a}$ acting on $\mathfrak{h}$ by derivations, we may form the **semidirect product** Lie algebra $\mathfrak{a} \ltimes \mathfrak{h}$ which is $\mathfrak{a} \oplus \mathfrak{h}$ as a vector space with commutator defined by

$$[(a_1, h_1), (a_2, h_2)] = ([a_1, a_2], a_1 \circ h_2 - a_2 \circ h_1 + [h_1, h_2]).$$

Note that a special case of this construction has already appeared in Example 15.23.

Theorem 16.7. *(Levi decomposition) If* $\mathrm{char}(\mathbf{k}) = 0$ *then we have* $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$*, where* $\mathfrak{g}_{\mathrm{ss}} \subset \mathfrak{g}$ *is a semisimple subalgebra (but not necessarily an ideal); i.e.,* $\mathfrak{g}$ *is isomorphic to the semidirect product* $\mathfrak{g}_{\mathrm{ss}} \ltimes \mathrm{rad}(\mathfrak{g})$*. In other words, the projection* $p : \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}}$ *admits an (in general, non-unique) splitting* $q : \mathfrak{g}_{\mathrm{ss}} \to \mathfrak{g}$*, i.e., a Lie algebra map such that* $p \circ q = \mathrm{Id}$.

Theorem 16.7 will be proved in Subsection 48.2.

Example 16.8. Let G be the group of motions of the Euclidean space $\mathbb{R}^3$ (generated by rotations and translations). Then $G = SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$, so $\mathfrak{g} = \mathrm{Lie}G = \mathfrak{so}_3(\mathbb{R}) \ltimes \mathbb{R}^3$, hence $\mathrm{rad}(\mathfrak{g}) = \mathbb{R}^3$ (abelian Lie algebra) and $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{so}_3(\mathbb{R})$.

Proposition 16.9. *Let* $\mathrm{char}(\mathbf{k}) = 0$*,* $\mathbf{k}$ *algebraically closed, and* V *be an irreducible representation of* $\mathfrak{g}$*. Then* $\mathrm{rad}(\mathfrak{g})$ *acts on* V *by scalars, and* $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ *by zero.*

Proof. By Lie's theorem, there is a nonzero $v \in V$ and $\lambda \in \mathrm{rad}(\mathfrak{g})^*$ such that $av = \lambda(a)v$ for $a \in \mathrm{rad}(\mathfrak{g})$. Let $x \in \mathfrak{g}$ and $\mathfrak{g}_x \subset \mathfrak{g}$ be the Lie subalgebra spanned by $\mathrm{rad}(\mathfrak{g})$ and x. Let W be the span of $x^n v$ for $n \geq 0$. By Lemma 15.18(i), W is a $\mathfrak{g}_x$-subrepresentation of V on which $a \in \mathrm{rad}(\mathfrak{g})$ has the only eigenvalue $\lambda(a)$. Thus by Lemma 15.18(iii), for $a \in \mathrm{rad}(\mathfrak{g})$ we have $\lambda([x, a]) = 0$, so the λ-eigenspace V_λ of $\mathrm{rad}(\mathfrak{g})$ in V is a $\mathfrak{g}$-subrepresentation of V, which implies that $V_\lambda = V$ since V is irreducible. □

Definition 16.10. $\mathfrak{g}$ is called **reductive** if $\mathrm{rad}(\mathfrak{g})$ coincides with the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$.

In other words, $\mathfrak{g}$ is reductive if $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$.

The Levi decomposition theorem implies that a reductive Lie algebra in characteristic zero is a direct sum of a semisimple Lie algebra and an abelian Lie algebra (its center). We will also prove this in Corollary 18.8.

16.2. **Invariant inner products.** Let B be a bilinear form on a Lie algebra $\mathfrak{g}$. Recall that B is invariant if $B([x, y], z) = B(x, [y, z])$ for any $x, y, z \in \mathfrak{g}$.

Example 16.11. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation of $\mathfrak{g}$ then the form

$$B_V(x, y) := \mathrm{Tr}(\rho(x)\rho(y))$$

is an invariant symmetric bilinear form on $\mathfrak{g}$. Indeed, the symmetry is obvious and

$$B_V([x, y], z) = B_V(x, [y, z]) = \mathrm{Tr}|_V(\rho(x)\rho(y)\rho(z) - \rho(x)\rho(z)\rho(y)).$$

Proposition 16.12. *If B is a symmetric invariant bilinear form on $\mathfrak{g}$ and $I \subset \mathfrak{g}$ is an ideal then the orthogonal complement $I^\perp \subset \mathfrak{g}$ is also an ideal. In particular, $\mathfrak{g}^\perp = \mathrm{Ker}(B)$ is an ideal in $\mathfrak{g}$.*

Exercise 16.13. Prove Proposition 16.12.

Proposition 16.14. *If B_V is nondegenerate for some V then $\mathfrak{g}$ is reductive.*

Proof. Let $V_1, ..., V_n$ be the simple composition factors of V; i.e., V has a filtration by subrepresentations such that $F_iV/F_{i-1}V = V_i$, $F_0V = 0$ and $F_nV = V$. Then $B_V(x, y) = \sum_i B_{V_i}(x, y)$. Now, if $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ then $x|_{V_i} = 0$, so $B_{V_i}(x, y) = 0$ for all $y \in \mathfrak{g}$, hence $B_V(x, y) = 0$. □

Example 16.15. It is clear that if $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{k})$ and $V = \mathbf{k}^n$ then the form B_V is nondegenerate, as $B_V(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}$. Thus $\mathfrak{g}$ is reductive. Also if n is not divisible by the characteristic of $\mathbf{k}$ then $\mathfrak{sl}_n(\mathbf{k})$ is semisimple, since it is orthogonal to scalars under B_V (hence reductive), and has trivial center. In fact, it is easy to show that in this case $\mathfrak{sl}_n(\mathbf{k})$ is a simple Lie algebra (another way to see that it is semisimple).

In fact, we have the following proposition.

Proposition 16.16. *All classical Lie algebras over $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ are reductive.*

Proof. Let $\mathfrak{g}$ be a classical Lie algebra and V its standard matrix representation. It is easy to check that the form B_V on $\mathfrak{g}$ is nondegenerate, which implies that $\mathfrak{g}$ is reductive. □

For example, the Lie algebras $\mathfrak{so}_n(\mathbb{K})$, $\mathfrak{sp}_{2n}(\mathbb{K})$, $\mathfrak{su}(p,q)$ have trivial center and therefore are semisimple.

16.3. **The Killing form and the Cartan criteria.**

Definition 16.17. The **Killing form** of a Lie algebra $\mathfrak{g}$ is the form $B_{\mathfrak{g}}(x,y) = \mathrm{Tr}(\mathrm{ad}x \cdot \mathrm{ad}y)$.

The Killing form is denoted by $K_{\mathfrak{g}}(x,y)$ or shortly by $K(x,y)$.

Theorem 16.18. *(Cartan criterion of solvability) A Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero is solvable if and only if $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Ker}(K)$.*

Theorem 16.19. *(Cartan criterion of semisimplicity) A Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero is semisimple if and only if its Killing form is nondegenerate.*

16.4. **Jordan decomposition.** To prove the Cartan criteria, we will use the Jordan decomposition of a square matrix. Let us recall it.

Proposition 16.20. *A square matrix $A \in \mathfrak{gl}_N(\mathbf{k})$ over a field $\mathbf{k}$ of characteristic zero can be uniquely written as $A_s + A_n$, where $A_s \in \mathfrak{gl}_N(\mathbf{k})$ is semisimple (i.e. diagonalizes over the algebraic closure of $\mathbf{k}$) and $A_n \in \mathfrak{gl}_N(\mathbf{k})$ is nilpotent in such a way that $A_sA_n = A_nA_s$. Moreover, $A_s = P(A)$ for some $P \in \mathbf{k}[x]$.*

Proof. By the Chinese remainder theorem, there exists a polynomial $P \in \overline{\mathbf{k}}[x]$ such that for every eigenvalue λ of A we have $P(x) = \lambda$ modulo $(x-\lambda)^N$, i.e.,

$$P(x) - \lambda = (x-\lambda)^N Q_\lambda(x)$$

for some polynomial Q_λ. Then on the generalized eigenspace $V(\lambda)$ for A, we have

$$P(A) - \lambda = (A-\lambda)^N Q_\lambda(A) = 0,$$

so $A_s := P(A)$ is semisimple and $A_n = A - P(A)$ is nilpotent, with $A_nA_s = A_sA_n$. If $A = A_s' + A_n'$ is another such decomposition then A_s', A_n' commute with A, hence with A_s and A_n. Also we have

$$A_s - A_s' = A_n' - A_n.$$

Thus this matrix is both semisimple and nilpotent, so it is zero. Finally, since A_s, A_n are unique, they are invariant under the Galois group of $\overline{\mathbf{k}}$ over $\mathbf{k}$ and therefore have entries in $\mathbf{k}$. □

Remark 16.21. 1. If $\mathbf{k}$ is algebraically closed, then A admits a basis in which it is upper triangular, and A_s is the diagonal part while A_n is the off-diagonal part of A.

2. Proposition 16.20 holds with the same proof in characteristic p if the field $\mathbf{k}$ is perfect, i.e., the Frobenius map $x \to x^p$ is surjective on $\mathbf{k}$. However, if $\mathbf{k}$ is not perfect, the proof fails: the fact that A_s, A_n are Galois invariant does not imply that their entries are in $\mathbf{k}$. Also the statement fails: if $\mathbf{k} = \mathbb{F}_p(t)$ and $Ae_i = e_{i+1}$ for $i = 1, .., p-1$ while $Ae_p = te_1$ then A has only one eigenvalue $t^{1/p}$, so $A_s = t^{1/p} \cdot \mathrm{Id}$, i.e., does not have entries in $\mathbf{k}$.

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Fall 2020