17. **Proofs of the Cartan criteria, properties of semisimple Lie algebras**

17.1. **Proof of the Cartan solvability criterion.** It is clear that $\mathfrak{g}$ is solvable if and only if so is $\mathfrak{g} \otimes_{\mathbf{k}} \overline{\mathbf{k}}$, so we may assume that $\mathbf{k}$ is algebraically closed.

For the "only if" part, note that by Lie's theorem, $\mathfrak{g}$ has a basis in which the operators $\mathrm{ad}x$, $x \in \mathfrak{g}$, are upper triangular. Then $[\mathfrak{g}, \mathfrak{g}]$ acts in this basis by strictly upper triangular matrices, so $K(x, y) = 0$ for $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$.

To prove the "if" part, let us prove the following lemma.

Lemma 17.1. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra such that for any $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$ we have $\mathrm{Tr}(xy) = 0$. Then $\mathfrak{g}$ is solvable.*

Proof. Let $x \in [\mathfrak{g}, \mathfrak{g}]$. Let $\lambda_i, i = 1, ..., m$, be the distinct eigenvalues of x. Let $E \subset \mathbf{k}$ be a $\mathbb{Q}$-span of λ_i. Let $b : E \to \mathbb{Q}$ be a linear functional. There exists an interpolation polynomial $Q \in \mathbf{k}[t]$ such that $Q(\lambda_i - \lambda_j) = b(\lambda_i - \lambda_j) = b(\lambda_i) - b(\lambda_j)$ for all i, j.

By Proposition 16.20, we can write x as $x = x_s + x_n$. Then the operator $\mathrm{ad}x_s$ is diagonalizable with eigenvalues $\lambda_i - \lambda_j$. So

$$Q(\mathrm{ad}x_s) = \mathrm{ad}b,$$

where $b : V \to V$ is the operator acting by $b(\lambda_j)$ on the generalized λ_j-eigenspace of x.

Also we have

$$\mathrm{ad}x = \mathrm{ad}x_s + \mathrm{ad}x_n$$

a sum of commuting semisimple and nilpotent operators. Thus

$$\mathrm{ad}x_s = (\mathrm{ad}x)_s = P(\mathrm{ad}x),$$

and $P(0) = 0$ since 0 is an eigenvalue of $\mathrm{ad}x$. Thus

$$\mathrm{ad}b = R(\mathrm{ad}x),$$

where $R(t) = Q(P(t))$ and $R(0) = 0$.

Let $x = \sum_j [y_j, z_j]$, $y_j, z_j \in \mathfrak{g}$, and d_j be the dimension of the generalized λ_j-eigenspace of x. Then

$$\sum_j d_j b(\lambda_j)\lambda_j = \mathrm{Tr}(bx) =$$

$$\mathrm{Tr}(\sum_j b[y_j, z_j]) = \mathrm{Tr}(\sum_j [b, y_j]z_j) = \mathrm{Tr}(\sum_j R(\mathrm{ad}x)(y_j)z_j).$$

Since $R(0) = 0$, we have $R(\mathrm{ad}x)(y_j) \in [\mathfrak{g}, \mathfrak{g}]$, so by assumption we get

$$\sum_j d_j b(\lambda_j)\lambda_j = 0.$$

Applying b, we get $\sum_j d_j b(\lambda_j)^2 = 0$. Thus $b(\lambda_j) = 0$ for all j. Hence $b = 0$, so $E = 0$.

Thus, the only eigenvalue of x is 0, i.e., x is nilpotent. But then by Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Thus $\mathfrak{g}$ is solvable. Thus proves the lemma. ☐

Now the "if" part of the Cartan solvability criterion follows easily by applying Lemma 17.1 to $V = \mathfrak{g}$ and replacing $\mathfrak{g}$ by the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$.

17.2. **Proof of the Cartan semisimplicity criterion.** Assume that $\mathfrak{g}$ is semisimple, and let $I = \mathrm{Ker}(K_\mathfrak{g})$, an ideal in $\mathfrak{g}$. Then $K_I = (K_\mathfrak{g})|_I = 0$. Thus by Cartan's solvability criterion I is solvable. Hence $I = 0$.

Conversely, suppose $K_\mathfrak{g}$ is nondegenerate. Then $\mathfrak{g}$ is reductive. Moreover, the center of $\mathfrak{g}$ is contained in the kernel of $K_\mathfrak{g}$, so it must be trivial. Thus $\mathfrak{g}$ is semisimple.

17.3. **Properties of semisimple Lie algebras.**

Proposition 17.2. *Let* $\mathrm{char}(\mathbf{k}) = 0$ *and* $\mathfrak{g}$ *be a finite dimensional Lie algebra over* $\mathbf{k}$. *Then* $\mathfrak{g}$ *is semisimple iff* $\mathfrak{g} \otimes_\mathbf{k} \overline{\mathbf{k}}$ *is semisimple.*

Proof. Immediately follows from Cartan's criterion of semisimplicity. Here is another proof (of the nontrivial direction): if $\mathfrak{g}$ is semisimple and I is a nonzero solvable ideal in $\mathfrak{g} \otimes_\mathbf{k} \overline{\mathbf{k}}$ then it has a finite Galois orbit $I_1, ..., I_n$ and $I_1 + ... + I_n$ is a Galois invariant solvable ideal, so it comes from a solvable ideal in $\mathfrak{g}$. ☐

Remark 17.3. This theorem fails if we replace the word "semisimple" by "simple": e.g., if $\mathfrak{g}$ is a simple complex Lie algebra regarded as a real Lie algebra then $\mathfrak{g}_\mathbb{C} \cong \mathfrak{g} \oplus \mathfrak{g}$ is semisimple but not simple.

Theorem 17.4. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $I \subset \mathfrak{g}$ *an ideal. Then there is an ideal* $J \subset \mathfrak{g}$ *such that* $\mathfrak{g} = I \oplus J$.

Proof. Let $I^\perp$ be the orthogonal complement of I with respect to the Killing form, an ideal in $\mathfrak{g}$. Consider the intersection $I \cap I^\perp$. It is an ideal in $\mathfrak{g}$ with the zero Killing form. Thus, by the Cartan solvability criterion, it is solvable. By definition of a semisimple Lie algebra, this means that $I \cap I^\perp = 0$, so we may take $J = I^\perp$. ☐

We will see below (in Proposition 17.7) that J is in fact unique and must equal $I^\perp$.

Corollary 17.5. *A Lie algebra $\mathfrak{g}$ is semisimple iff it is a direct sum of simple Lie algebras.*

Proof. We have already shown that a direct sum of simple Lie algebras is semisimple. The opposite direction easily follows by induction from Theorem 17.4. □

Corollary 17.6. *If $\mathfrak{g}$ is a semisimple Lie algebra, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof. For a simple Lie algebra it is clear because $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$ which cannot be zero (otherwise, $\mathfrak{g}$ would be abelian). So the result follows from Corollary 17.5. □

Proposition 17.7. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ be a semisimple Lie algebra, with $\mathfrak{g}_i$ being simple. Then any ideal I in $\mathfrak{g}$ is of the form $I = \oplus_{i \in S} \mathfrak{g}_i$ for some subset $S \subset \{1, ..., k\}$.*

Proof. The proof goes by induction in k. Let $p_k : \mathfrak{g} \to \mathfrak{g}_k$ be the projection. Consider $p_k(I) \subset \mathfrak{g}_k$. Since $\mathfrak{g}_k$ is simple, either $p_k(I) = 0$, in which case $I \subset \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_{k-1}$ and we can use the induction assumption, or $p_k(I) = \mathfrak{g}_k$. Then $[\mathfrak{g}_k, I] = [\mathfrak{g}_k, p_k(I)] = \mathfrak{g}_k$. Since I is an ideal, $I \supset \mathfrak{g}_k$, so $I = I' \oplus \mathfrak{g}_k$ for some subspace $I' \subset \mathfrak{g}_1 \oplus \quad \oplus \mathfrak{g}_{k-1}$. It is immediate that then I' is an ideal in $\mathfrak{g}_1 \oplus \quad \oplus \mathfrak{g}_{k-1}$ and the result again follows from the induction assumption. □

Corollary 17.8. *Any ideal in a semisimple Lie algebra is semisimple. Also, any quotient of a semisimple Lie algebra is semisimple.*

Let $\mathrm{Der}\mathfrak{g}$ be the Lie algebra of derivations of a Lie algebra $\mathfrak{g}$. We have a homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}\mathfrak{g}$ whose kernel is the center $\mathfrak{z}(\mathfrak{g})$. Thus if $\mathfrak{g}$ has trivial center (e.g., is semisimple) then the map ad is injective and identifies $\mathfrak{g}$ with a Lie subalgebra of $\mathrm{Der}\mathfrak{g}$. Moreover, for $d \in \mathrm{Der}\mathfrak{g}$ and $x \in \mathfrak{g}$, we have

$$[d, \mathrm{ad}x](y) = d[x, y] - [x, dy] = [dx, y] = \mathrm{ad}(dx)(y).$$

Thus $\mathfrak{g} \subset \mathrm{Der}\mathfrak{g}$ is an ideal.

Proposition 17.9. *If $\mathfrak{g}$ is semisimple then $\mathfrak{g} = \mathrm{Der}\mathfrak{g}$.*

Proof. Consider the invariant symmetric bilinear form

$$K(a, b) = \mathrm{Tr}|_{\mathfrak{g}}(ab)$$

on $\mathrm{Der}\mathfrak{g}$. This is an extension of the Killing form of $\mathfrak{g}$ to $\mathrm{Der}\mathfrak{g}$, so its restriction to $\mathfrak{g}$ is nondegenerate. Let $I = \mathfrak{g}^{\perp}$ be the orthogonal complement of $\mathfrak{g}$ in $\mathrm{Der}\mathfrak{g}$ under K. It follows that I is an ideal, $I \cap \mathfrak{g} = 0$, and $I \oplus \mathfrak{g} = \mathrm{Der}\mathfrak{g}$. Since both I and $\mathfrak{g}$ are ideals, we have $[\mathfrak{g}, I] = 0$. Thus for $d \in I$ and $x \in \mathfrak{g}$, $[d, \mathrm{ad}x] = \mathrm{ad}(dx) = 0$, so dx belongs to the center of $\mathfrak{g}$. Thus $dx = 0$, i.e., $d = 0$. It follows that $I = 0$, as claimed. □

Corollary 17.10. *Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra, and $G = \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$. Then G is a Lie group with $\mathrm{Lie}G = \mathfrak{g}$. Thus G acts on $\mathfrak{g}$ by the adjoint action.*

Proof. It is easy to show that for any finite dimensional real or complex Lie algebra $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$, so the statement follows from Proposition 17.9. □